Attachment No. 3

Author's abstract

1. First name and surname.

Andrzej Wal

2. Diplomas obtained, academic degrees – including the title, place and year of issue

1988	Master of Science degree in physics obtained at the Faculty of Mathematics and Physics of the Pedagogical University in Rzeszów. Title of the master's thesis: "Impact of external conditions on selective reflection in cholesteric liquid crystals"; promoter: dr. hab. Henryk Konwent.
1997	Doctorate degree in physical science obtained at the Faculty of Physics of Adam Mickiewicz University in Poznań. Title of the doctoral thesis: "Crystallographic and magnetic gauge symmetries of a finite linear chain"; promoter: Prof. Tadeusz Lulek, Adam Mickiewicz University in Poznań; advisers: Prof. Jacek Karwowski, Nicolaus Copernicus University in Toruń; Prof. Wiesława Sikora, AGH University of Science and Technology .

3. Information concerning employment in scientific institutions

1987	Assistant (probationer) at Solid States Physics Unit at the Institute of Physics (ZFCS IF) at the Faculty of Mathematics and Physics of the Pedagogical University (WSP) in Rzeszów
1988-1997	Assistant at Solid States Physics Unit at the Institute of Physics (ZFCS IF) at the Faculty of Mathematics and Physics of the Pedagogical University (WSP) in Rzeszów
1997-2007	Assistant professor at ZFCS IF at the Faculty of Mathematics and Natural Sciences of the University of Rzeszow (UR)
2007-2010	Senior lecturer at ZFCS IF at the Faculty of Mathematics and Natural Sciences of the University of Rzeszow (UR)
2011-2013	Assistant professor at ZFCS IF at the Faculty of Mathematics and Natural Sciences of the University of Rzeszow (UR)
Since 2013	Senior lecturer at the Faculty of Mathematics and Natural Sciences of the University of Rzeszow (UR)

4. Indication of achievements[1] pursuant to article 16 item 2 of the act of law dated 14 March 2003, concerning scientific degrees and scientific titles, and degrees and titles in the domain of arts (Official Journal No. 65, item 595, with later amendments):

According to the aforementioned act of law, I wish to indicate the single subject cycle of 8 publications. The papers sum up my research since 2005 and form self-coherent integrity devoted to description of electron in two-dimensional lattice with Born-Karman border conditions and subjected to quantised magnetic field.
I present publications mentioned above in the chronological order, providing for each of them the Journal impact factor and the number of quotations (according to the Web of Science database) in the footnote.

a) Title of scientific/artistic achievement

"The band structure of Bloch electrons on a finite two-dimensional system in a quantized magnetic field"

b) Publications concerning the scientific achievement; listed in chronological order (author/ authors, title/ titles of publication, year of publication, name of the publishing house)

H1. A. Wal, *Magnetic translation group for a finite system*, 2005, Physica Status Solidi (b), **242**, 291–295.
impact factor: 0.836; number of quotations: 2

H2. A. Wal, *Tight binding analogue of cyclotron orbits*, 2007, Physica Status Solidi (b), **244**, 2559–2563.
impact factor: 1.071; number of quotations: 1

H3. A. Wal, *The magnetic translation group for a finite system and the Born-Karman boundary condition*, 2008, Journal of Physics: Conference Series, **104**, 012021.
impact factor: none ; number of quotations:2

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[1] In case of achievements accomplished by means of joint work/ works, declarations of all its/ their co-authors should be presented, stating their estimated percentage individual contributions

H4. A. Wal, *The structure of magnetic translation group for a finite system*, 2009, Physica B-Condensed Matter, **404**, 1040–1044.
impact factor: 1.056 ; number of quotations: 3

H5. A. Wal, *Multielectron irreducible representations of the magnetic translation group*, 2009, Physica Status Solidi (b), **246**, 1024–1028.
impact factor: 1.150 ; number of quotations: 1

H6. A. Wal, *The symmetry of three-electron states in a quantized magnetic field*, 2011, Physica B: Condensed Matter, **406**, 2734–2739.
impact factor: 1.063; number of quotations: 1

H7. A. Wal, *Band structure, Brillouin zone, and condensation of states for an itinerant electron in a magnetic quantum dot*, 2013, Physica B: Condensed Matter, **410**, 222–226.
impact factor: 1.327; number of quotations: 1

H8. A. Wal, *Energy bands for finite two-dimensional systems in a quantized magnetic field: the symmetry of the model*, 2013, Journal of Mathematical Chemistry, **51**, 2285–2316.
impact factor:1.226; number of quotations: 0

c) description of scientific / artistic objective of the above work / works and accomplished results, including description of their possible use

The scientific aim of the work is the description of properties of an electron in a two-dimensional, finite lattice in a quantized magnetic field, especially the change in the structure of a finite analogon of energy bands and in the Brillouin zone of such system, as well as determination of quantum numbers which characterize states, also the multi-electron ones. The method of description is based on tools provided by the symmetry and group theory. These systems reveal so interesting physical properties that they were, for a long time, an object of interest for experimentalists as well as theoreticians. This interest has grown rapidly after discovery of an integer and fractional Hall effect, and attempts for their theoretical explanation. The characteristic behavior of such systems consists in dependence of band structure on, even very small, change of quantized magnetic field which is manifested by the "Hofstadter butterfly"[2] on the graph $E(\eta)$, with η being the number of quanta of magnetic flux passing through the

[2] D. Hofstadter, Phys. Rev. B, 14(6), 2239–2249 (1976).

elementary lattice cell. The spectrum is of fractal shape, i.e. energy levels for a small change of flux are similar to those, which can be seen at the large scale. For irrational number η the spectrum reveals properties of Cantor set. These effects are related to quantization of a magnetic field, strictly speaking, they are a result of incommensurance of number of quanta of flux and sizes of elementary cell of a two-dimensional lattice.

Despite the long term of investigations for such systems, still there appear new interesting problems concerning the subject. They are related to theoretical problems like: composite fermions[3], quantum groups and applications of Bethe Ansatz for two-dimensional systems[4], quantization problems for low-dimensional manifolds[5], and usage of boundary conditions for a finite lattice[6]. Within last years such a topic has become more important, because it is possible now to achieve so large magnetic flux penetrating the cell of the lattice, that one can observe fractal character of the spectrum. This accomplishment is realized by the usage of optical lattices, which allow to simulate a magnetic field, with the magnitude adequate to notice such effects[7]. Last year, there came out the paper in which authors informed about observation of fractal character of the spectrum with the use of Moire superlattice[8]. The lattice was built by placing atomic layer of graphene on the hexagonal boron nitride (BN).

The subject presented in this cycle of papers is connected first of all with application of the symmetry to description of characteristic features of models being considered. In the literature concerning this topic one can find the mathematical construction called magnetic translation group (MTG), which replaces the usual translation group in the presence of a quantized magnetic field. It was introduced by Zak and Brown in the '60s of the last century[9]. This group provides, by its irreducible representations, the description of states of an electron on a two-dimensional crystal lattice subjected to a perpendicular magnetic field with a quantized flux. The quantization is tuned in such a way, that a flux passing through the elementary cell is

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[3] J. J Quinn, A. Wojs, K.-S. Yi, G. Simion, Phys. Rep., 481(3-4), 29–81 (2009).

[4] P. B. Wiegmann, A. V. Zabrodin, Mod. Phys. Lett. B, 08(05), 311–318. (1994); Y. Hatsugai, M. Kohmoto, Y.-S. Wu, Phys. Rev. B, 53(15), 9697–9712 (1996).

[5] R. Alicki, J. Klauder, J. Lewandowski, Phys. Rev. A 48, 2538–2548 (1993); K. Kowalski, J. Rembieliński, J. Phys. A. Math. Gen., 38, 8247–8258 (2005).

[6] K. Czajka, A. Gorczyca, M. M. Maska, M. Mierzejewski, Phys. Rev. B, 74, 125116 (2006).

[7] M. Aidelsburger, et. al., Phys. Rev. Lett. 107, 255301 (2011); K. Jimenez-Garcia, et. al., Phys. Rev. Lett. 108, 225303 (2012).

[8] C. R. Dean, et. al., Nature, 497, 598, (2013).

[9] J. Zak, Phys. Rev., 134, A1602–A1606 i A1607–A1611, 1964; E. Brown, Phys. Rev., 133, A1038–A1044, (1964).

given by a fraction $\eta = p/q$ in units of flux quantum h/e, where p and q are integers with no common divisor. This attempt demands to take into account mutual relations between quantization of a magnetic field, the size of the finite lattice and applied boundary conditions. The magnetic field is described by its vector potential, which can be changed by the choice of appropriate gauge. The selection of a gauge doesn't change the energy of the system, however it can be important taking into account the compatibility conditions between parameters of magnetic field and chosen boundary conditions.

The group theory used for systems being investigated allows to analyze problems concerning the subject, seldom discussed till now in the literature. The most important I would like to list are: dependence of the structure of MTG on the chosen gauge and applied boundary conditions, determination of equivalence relations in the set of irreducible representations whose parameters provide exact quantum numbers characterizing band structure of finite crystals, description of a change of the Brillouine zone in a quantized magnetic field, and presentation of "non-physical" representations, with the way of their determination and interpretation in the context of multi-electron systems.

My research was restricted mainly to the systems of a finite size. For these structures the Born-Karman boundary conditions were applied, since I was interested in the description of energy bands as the function of quasimomentum, like it is usually presented in condensed matter physics. There is another attempt related to Haldane sphere, where the main parameter is angular momentum, however, I prefer to use the symmetry given by the magnetic translation group what determines the choice of periodic boundary conditions.

At the beginning my research concerned compatibility conditions between magnetic field given by the vector potential together with the quantization parameter $\eta = p/q$ and sizes of the finite lattice. These conditions were described with the help of the magnetic translation group and its irreducible representations. The latter were determined with the use of induction procedure, which allows computer implementation of an algorithm for evaluation of irreducible representations for given parameters of the model. The important result in this part of research was demonstration that boundary conditions applied to finite systems determine the type of the vector potential being applied, since both should satisfy compatibility relations. I considered two the most popular gauges: Landau and symmetric one.

Compatibility relations are associated also with the dependence of the structure of MTG on the chosen gauge. Considering both of the mentioned gauges, I showed how the details of this structure can be determined by the gauge. At this stage of research I was able to present each group element as a product of generators.

In the literature related to the discussed subject occurs the term "magnetic Brillouin zone"[10], which illustrates changes in the Brillouin zone of two-dimensional crystal subjected to a quantized magnetic field. Generally, these changes are described through reduction of possible values of quasimomenta along the chosen direction in a reciprocal space. The detailed analysis of irreducible representations of MTG, whose parameters define non-equivalent quasimomenta, allows to determine "magnetic band Brillouin zone", as a set of wave vectors $\boldsymbol{k}$ over which finite analogues of energy bands are defined.

Zak in the paper introducing MTG, defined selected irreducible representations as "physical", i.e. the ones which are important in the description of one-electron states. They are determined by choosing a given value of a parameter of irreducible representations of MTG, strictly speaking, representations of a gauge group being a subgroup of MTG. The other values of this parameter are described as non-physical. Florek in his paper introduced an idea that representations related to the latter parameters can be applied to the multi-electron system[11]. I extended this approach to the system of identical particles taking into account the symmetric group (non-distinguishable particles) as well as the Hubbard type interactions between electrons.

The cycle of papers begins with the presentation of methods of determination of MTG and its irreducible representations for finite lattices. In the following works I studied properties of such a system subjected to a quantized magnetic field. The main results of investigations are listed below:

1. Determination of the structure of the magnetic translation group for selected gauges of a magnetic field (Landau and symmetric). Elements of MTG can be presented in the form $(\boldsymbol{t}, \gamma)$, where $\boldsymbol{t}$ means a translation and γ is a phase related to the translation (effect of a magnetic field). Calculations show, that a set of phases connected with a translation depends not only on the chosen gauge and the value of the parameter $\eta = p/q$, but also on arithmetic properties of components of the vector $\boldsymbol{t}$. The latter dependence is observed, however, only for the symmetric gauge. It was shown also, that for such a gauge the order of MTG does not depend on the parity of integer p, however, the parity of this integer influences on the detailed structure of the group.

[10] M. Kohmoto, Ann. Phys. (N. Y), **160**, 343–354 (1985).

[11] W. Florek, Phys. Rev. B, **55**, 1449–1453, (1997); J. Math. Phys. **42**, 5177, (2001).

2. Determination of relations between parameters of a quantized magnetic field, Born-Karman boundary conditions and the chosen gauge. Analysis of this type of dependences shows that for symmetric gauge, in order to use periodic boundary conditions, it is necessary to restrict possible values of quantization of magnetic field to even values of integer p. For Landau gauge this type of reliance is not observed, i.e. all values of $p = 1,2,3, \dots L$ are allowed, where L is the size of a lattice in a given direction. The main result of this part of research is determination of influence of the chosen gauge on the admissible quantization of a magnetic field, when periodic boundary conditions are applied.
3. Interpretations of "non-physical" irreducible representations $\Gamma^{\kappa_x \kappa_y s}$ of magnetic translation group. In the literature on this subject, one can use usually "physical" representations, i.e. representations selected by the choice of the parameter $s = 1$. The integer s labels representations of a subgroup of MTG related to the phase factor γ, i.e. the gauge group. The way of determination of irreducible representations for larger values of s was presented, and physical interpretation of them was introduced. These representations can be used for the description of multi-electron systems. The characterization of such systems should contain (because of indistinguishability of particles) the symmetric and the unitary group, introduced according to the duality of Weyl[12]. The considered model was supplemented by the interaction of the Hubbard type.
4. Determination of parameters describing the band structure in a magnetic field. Parameters κ_x, κ_y of irreducible representation $\Gamma^{\kappa_x \kappa_y s}$ of MTG label band states. Values of these quasimomenta create magnetic Brillouin zone (MBZ). Quantum number γ', being the dimension of representation $\Gamma^{\kappa_x \kappa_y s}$, labels subbands on which the non-degenerate band splits in a magnetic field. For a magnetic field described by a fraction $\eta = p/q$, this dimension is equal to $\gamma' = q$. The second parameter γ determines the degree of degeneration of each subband and is related to the multiplicity of representation Γ in decomposition of reducible representation of MTG, defined in the positional space. Finally, a single non-degenerate band of a finite crystal without magnetic field splits in a quantized field into q subbands, each of them being q-tuply degenerate. Additional quantum number s related to gauge subgroup of MTG describes the number of electrons in the system.

The magnetic field changes the translational symmetry of the system introducing non-abelian type of elements of the symmetry group. This non-commutativity

[12] H. Weyl, *The theory of groups and quantum mechanics*, New York: Dover Publications (1984).

leads indeed to rising up the symmetry, what is reflected in the multidimensionality of irreducible representations of MTG. It stays in the contrast with the one-dimensional Bloch states being the representations of an abelian translation group. Symmetry of the new structure in a magnetic field is described by a magnetic cell, which is q–times larger than the crystallographic one. Magnetic Brillouin zone is q–tuply rarefied, i.e. consists of smaller set of admissible quasimomenta in a given direction, say y. It was shown, that this rarefaction should be supplemented by complementary reduction of quasimomenta in the direction perpendicular to the mentioned one. It results from the equivalent relation in the set of irreducible representations. The band structure defined over this rarefied zone, called magnetic band Brillouin zone (MBBZ), illustrates the process of condensation of states over admissible quasimomenta, under symmetry of MTG. Such a degeneration together with a rarefaction of the zone are analogues of merging of free electron bands in a magnetic field into Landau levels.

5. Description of induction method of irreducible representations of magnetic translation group for multi-electron systems. The number of electrons is related to the parameter s of irreducible representations $\Gamma^{\kappa_x,\kappa_y,s}$. It was shown, that determination of such representations for $s > 1$ is relatively easy in the case, when this parameter has no common divisor with parameter q describing magnetic field (the magnetic flux is defined by $\eta = p/q$).
6. Determination of transition matrices between three bases: position, momentum and symmetry adapted ones. They allow a full characterization of states similar to the Racah theory of angular momentum adapted to finite crystals. This is true for one-electron as well as for multi-electron states, however, quantum numbers of the latter concern physical quantity related to the whole system of s electrons. In the construction of transition matrices the projection operators, built with the help of irreducible representations of MTG and symmetric groups Σ_s, were used.

Knowledge of energy spectrum for nano and mesoscopic systems is important nowadays on account of technical possibility of producing such structures. The topics presented in the cycle of papers can be applied in description of discussed structures subjected to magnetic field. In this context, particularly interesting is the determination of physically correct and mathematically treated boundary conditions applied to the finite lattice. Equally interesting and instructive is the study of changes in energy band structure, caused by the quantized magnetic field and finiteness of the lattice. An acquired experience will allow to design a system of given energy spectrum[13].

Studies conducted in the works H1–H8

H1. A. Wal, *Magnetic translation group for a finite system*, 2005, Physica Status Solidi (b), **242**, 291–295.

In this paper the model of an itinerant electron was considered, with interaction restricted to the nearest neighbors only. The dynamics of the system was described by the tight binding Hamiltonian and the finite crystal was closed with the Born-Karman boundary conditions. The simplest model with isotropic hopping integral was used and magnetic field was introduces by phase $\vartheta_{\lambda\mu}$ defined for each edge (λ,μ) of lattice cells, λ and μ meaning the adjacent nodes of the crystal lattice. The key parameter of this model is fraction $\eta = p/q$ representing the magnetic flux passing through elementary cell of the lattice and expressed in units of flux of magnetic quantum h/e. The symmetric gauge was chosen for the field.

The symmetry of this model was described by magnetic translation group. This group for a finite two-dimensional crystal (on the example with spatial size $L_x \times L_y = 4 \times 4$ and $\eta = 1/2$, where the sizes of a lattice were presented in a unit of length of the elementary cell) was calculated and characterized. The method of determination of irreducible representations of MTG, based on the induction procedure from a maximal subgroup, was presented[14]. This subgroup is chosen according to the magnetic cell, i.e. the cell for which the summation of the phases along its edges results in an integer number. This means that the magnetic flux through such a cell is integer multiplicity of flux quantum. The subgroup is abelian, which is a helpful feature to find its irreducible representations.

The method of induction has proved to be effective, and thus was written as an algorithm. It was implemented in the Maple package and used in further calculations for the purpose of subsequent papers concerning the subject.

The parameters of resulting irreducible representations label electron states in both magnetic, and periodic potential. In the context of following papers it should be emphasized, that periodic boundary conditions were applied. As it will be shown they have an influence on compatibility conditions, i.e. on the relation between the size of the lattice, the quantization of magnetic field, and the used gauge.

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[13] J. Zak, J. Phys. Conf. Ser. **104**, 012013 (2008).

[14] J. Mozrzymas, Application of group theory in physics, PWN Warszawa, Wrocław, 1977 (in Polish).

H2. A. Wal, *Tight binding analogue of cyclotron orbits*, 2007, Physica Status Solidi (b) **244**, 2559–2563.
In the paper, a tight binding analogue of cyclotron orbit were considered, as well as relations between boundary conditions and quantization of magnetic field. The Born-Karman boundary conditions were applied. They introduced quasimomentum, the quantum number widely used in the description of energy bands of crystals. The tight binding model expressed by hopping Hamiltonian describes the interactions of electrons with a magnetic field and a periodic potential. The structure of energy bands is very sensitive to the change of η, which leads to the dependence of the type of Hofstadter butterfly.

The gauge was chosen as the symmetric one. The boundary conditions provide the equivalence relations between elements of magnetic translation group. This leads at the same time to restriction of possible quantization of magnetic field, i.e. we obtain the relation $\eta = r/L$, where L is a period of two-dimensional crystal and r is an even number.

The Hamiltonian should commute with elements of the magnetic translation group, in particular with the ones, which concern the translation by primitive vectors of the lattice. This condition provides the distributions of a phase (depending on the chosen gauge) defined for each edge of elementary cells. In the paper, such distribution was presented for the finite crystal. The spreading of the phases should be consistent with boundary conditions applied, which limits the permissible values of fraction η. Its nominator should belong to the set of even integers from the range $r \in \{0, L^*\}$, where L^* is an even number smaller than L.

Operators $T(a)$ and $T(b)$ introduced in the paper represent the action of the magnetic translation group on the space of quantum states of the electron. They are unitary noncommuting operators and we are not able to diagonalize two of them simultaneously. The basis of states can be chosen in such a way, that one of operators, say $T(b)$ can be diagonalized. Then the second, $T(a)$, cannot be fully diagonalized, and each irreducible subspace is a natural analogue of the cyclotronic orbit of the free electron case. $T(a)$ acts in a way cyclically in each irreducible subspace, what completes the tight binding analogy to cyclotron orbits.

H3. A. Wal, *The magnetic translation group for a finite system and the Born-Karman boundary condition*, 2008, Journal of Physics: Conference Series, **104**, 012021.
In the paper, the discussion mentioned in paper H2 and concerning mutual relations between Born-Karman conditions and two types of applied gauge, i.e. symmetric and

Landau, was extended. The main aim of research was assessment of usefulness of both gauges in the description of finite systems with periodic boundary conditions.

Elements of magnetic translation group can be written as $(\boldsymbol{t},\gamma)$, where $\boldsymbol{t}$ is a translation belonging to the translation group T of the finite crystal and γ is a phase introduced by a magnetic field. The phase part, and thus also, group multiplication, depends on the chosen gauge. Periodic boundary conditions establish equivalence relations between group elements, which differ by a period N of the finite crystal (the size of the crystal is given in unit of elementary cell). These relations together with the multiplication rule determine criteria for quantization of a magnetic field, i.e. admissible value of η. For Landau gauge this parameter should be in the form $\eta = n/N$, where $n = 0, 1, \dots N-1$. For the symmetric gauge, however, numerator of the fraction η should be an even integer from the range $(0, N-1)$. Quantization of the magnetic flux within considered model depends, in this way, on the chosen gauge. From two presented gauges, the Landau gauge is better adapted to the symmetry of the model, since, in this case, the set of admissible values of η is two times larger than for the symmetric gauge. The conclusion is, that for a small system the mutual relation between sizes of the lattice, chosen boundary conditions and gauges are important and should be taken into account in the description of such systems.

H4. A. Wal, *The structure of magnetic translation group for a finite system*, 2009, Physica B-Condensed Matter, **404**, 1040–1044.

The main aim of the paper was determination of the structure of the magnetic translation group and examination how it depends on the chosen gauge of a magnetic field. The influence of vector potential on the features of considered model was observed already in the previous papers (H2-H3), where an interrelationship between the sizes of the lattice, quantization of flux and applied gauge was noticed. This time, particular emphasis was put on the investigation of dependence of the structure on the vector potential of a magnetic field. Research was carried out on the rectangular finite lattice with a magnetic field defined by the parameter $\eta = p/q$, where p and q are integers with no common divisors.

Elements of magnetic translation group consist of a translation $\boldsymbol{t}$ and an associated phase, denoted as γ. This phase is determined by the chosen gauge and the parameter η. Properties of elements of MTG were investigated for the symmetric and the Landau gauge.

For the symmetric gauge, phase factor depends on the magnetic field according to the formula $\gamma = \exp(2\,\pi i \frac{1}{2}\frac{p}{q}\theta)$, where $\theta \in \{0,1,\dots q\}$ for even p and $\theta \in \{0,1,\dots 2q\}$ for odd p[15]. In this way, it appears at the first sight, that the order of the MTG, which can be calculated as a product of the order of the translation group T by the number of elements in the set of θ, changes according to the parity of p. However, the careful calculations didn't confirm this hypothesis. In order to establish the relation between a translation $\boldsymbol{t}$ and a corresponding phase γ, both of them forming an element of MTG, the multiplication rule for the group was used. The gathering of the phase during the movement of the electron is represented in the considered approach through the multiplication of group elements expressed by group generators. Calculations showed, that the order of MTG doesn't depend on the parity of p and is equal to $|\mathrm{T}|q$, where $|\mathrm{T}|$ means the order of the translation group. It should be stressed, however, that details of the structure of MTG exhibit dependence on parity of this integer. For an even value, each translation is accompanied with the same set of phases, while for an odd p the number of phases in the set is the same but their values depend on the components (t_x, t_y) of the vector $\boldsymbol{t}$.

Similar calculations were performed for the Landau gauge. It appeared that the order of a group is the same, what is expected for an isomorphic group representing the same abstract group. This time, however, the set of phases associated with each translation is the same, independent on the parity of p.

For both considered gauges, each group element was expressed by two generators introduced in this paper. The features of maximal abelian group and the class of conjugated elements were also provided. These structures are important for determination of irreducible representations of MTG. Results obtained in this paper were the basis for further investigations related to the energy band structure in a magnetic field.

H5. A. Wal, *Multielectron irreducible representations of the magnetic translation group*, 2009, Physica Status Solidi (B), **246** , 1024–1028.

In the paper, the procedure of determination of irreducible representations $\Gamma^{\kappa_x,\kappa_y,s}$ of the magnetic translation group was presented for the parameter $s = 1$, as well as for $s > 1$. The procedure is based on the method of induction from the maximal abelian subgroup, because its one-dimensional representations may be easily obtained. Usually in the literature the so called "physical" representations (parameter $s = 1$) are discussed. They are used for the description of one-electron states. Representations with other value of s are called "nonphysical". There are papers by Florek[16], in which the

[15] W. Opechowski, W. Tam, Physica **42**, 529–556 (1969).

[16] W. Florek, *Phys. Rev. B* **55**, 1449–1453 (1997), W. Florek, J. *Phys. Condens. Matter* **11**, 2523–2529 (1999).

author shows, with the help of product of irreducible representations Γ, that representations with $s = 2$ can be applied for the description of two-electron states. In the present paper it was demonstrated that reducible representations, being the tensor product of two reducible representations related to one electron state, decompose into irreducible representations of MTG with the parameter $s = 2$. It means, that the parameter is connected with the number of electrons in the system and irreducible representations with such a value of s can be used for the description of multi-electron states. In the decomposition of tensor product of reducible representations symmetric and anti-symmetric parts were considered in order to find anti-symmetric states, according to the Pauli exclusion principle. This allowed to consider the spin in the model.

The induction procedure is simple for case $s = 1$ but it can be easily extended also for a larger value of the parameter. In the paper it was shown that an outline of the induction remains the same, for those values of s, which have no common divisors with q, the quantity characterizing the magnetic field through parameter $\eta = p/q$. The method of induction was illustrated on an example of a finite lattice with quantization of magnetic field given by $\eta = 1/3$.

H6. A. Wal, *The symmetry of three-electron states in a quantized magnetic field*, 2011, Physica B: Condensed Matter, **406**, 2734–2739.
The main aim of the paper was presentation of kinematics of multi-electron states for a two-dimensional square lattice subjected to quantized magnetic field. Multi-electron states were expressed with the use of tensor products of one-electron states and the main classification tool was the symmetry. In order to obtain the full description the spin was also considered, what caused the use of permutation and unitary symmetry together with the translational one. The symmetry was described by corresponding groups: the magnetic translation group, the symmetric group (permutation of electrons) and the unitary group (permutation of electron states). The introduction of such symmetries to the description of the system, allows to distinguish orbital and spin part in the space of states in the suitable tensor product representing multi-electron states. This separation permits to restrict discussion only to "orbital" (spatial) part, because its symmetry determines the symmetry of the "spin" part with which the former is connected, in order to guarantee the anti-symmetrical property of the whole wave function of electrons.

Irreducible representations of mentioned groups were used in determination of symmetry adapted bases, according to the duality of Weyl. Representations of the permutation Δ^{λ} and the unitary D^{λ} group are characterized by partitions λ of number of electrons n (i.e. by the decompositions of number n into the sum of integers). Representations of magnetic translation group $\Gamma^{\kappa_x,\kappa_y,s}$ are labeled by quasimomenta κ_x, κ_y (these symbols are used to distinguish them from the quasimomenta $(k_x, k_y) = (2\pi\kappa_x, 2\pi\kappa_y)$) and s. The last parameter is connected with the representation of phase subgroups, but at the same time is related to the number n of electrons in considered system. This means that the representation describing the translational symmetry of n electrons in a magnetic field is of the form $\Gamma^{\kappa_x,\kappa_y,n}$. The tensor product of representations $M^{\otimes n}$ (M means the representation of the magnetic translational group defined in the space of one-electron states) acting on positional space of n electrons, decomposes into irreducible representations $\Gamma^{\kappa_x,\kappa_y,n}$.

The irreducible representations of discussed symmetry groups allow to determine bases adapted to the translational symmetry in a magnetic field, as well as to the permutational one. The former is given by parameters (κ_x, κ_y, s) of suitable irreducible representation and by additional parameter β related to multiplicity of representation $\Gamma^{\kappa_x,\kappa_y,s}$ in the decomposition of $M^{\otimes n}$. The basis calculated with the use of projection operator can be written in the form $b_{irr}^{MTG} = \{|\kappa_x, \kappa_y, s, \beta\rangle\}$.

The basis $b_{irr}^{o} = \{|\lambda, t, y\rangle\}$ adapted to the permutational symmetry is determined by the partition λ of the integer n, as well as by bases of irreducible representations $t \in D^{\lambda}$ and $y \in \Delta^{\lambda}$ of unitary and symmetric group, respectively. This basis should be connected with suitable basis $b_{irr}^{s} = \{|\bar{\lambda}, y\rangle\}$ of a spin subspace, where $\bar{\lambda}$ means transposition of partition λ, and y is a semistandard Yang tableaux. The basis of "orbital" part was determined with the use of projection operator, which selects from products of one-electron states only those, which exhibit appropriate symmetry under permutation of particles.

Irreducible bases (adapted to the symmetry of the system) obtained in this way, allow to construct the matrices which are used to transform the positional basis $b = \{|i, j\rangle\}$, where i, j mean the coordinates of an electron on the square lattice, to the symmetry adapted.

The energy of electrons is determined with the help of tight binding Hamiltonian, but extended by the Hubbard term, responsible for repulsive interaction between electrons with the same node coordinates, but with a different spin projection. The Hamiltonian can be diagonalized by transformation from the positional to the symmetry adapted basis. When we use the translational symmetry, quantum numbers are quasimomenta, but in the case of permutation the quantum number is the total spin S. Thanks to the matrices describing the transformation between different bases it is possible to determine for each state the value of quasimomentum (and this way, to find out the energy band $E(\boldsymbol{k})$) as well as the total spin S. In the paper, the Zeeman's effect was not considered, however it is possible to include it within the model being discussed.

Calculations were carried out for the case of three electrons. This choice was determined, first of all, by the need to provide clearness of description, which can be destroyed by the "combinatorial explosion" arising for larger value of n. The method was explained in details on the example of the finite system of the size 3×3 subjected to a quantized magnetic field given by $\eta = 1/3$. This model is rather small but effectively solved and is sufficient to demonstrate main features of the physical system.

H7. A. Wal, *Band structure, Brillouin zone, and condensation of states for an itinerant electron in a magnetic quantum dot*, 2013, Physica B: Condensed Matter, **410**, 222–226.
The main aim of the paper was the description of the band structure, in the tight binding approximation, of two-dimensional, finite lattice subjected to a quantized magnetic field. The Born-Karman boundary conditions were applied and the energy was determined in the unit of hopping integral t, which describes the interaction of the nearest neighbors (interactions between next neighbors were omitted). Usually the band structure is determined over the Brillouin zone (BZ) defined by irreducible representations of the translational group being the symmetry group of two-dimensional crystal. The presence of a magnetic field changes the translational symmetry introducing instead non-abelian symmetry given by the magnetic translation group. There is an abelian subgroup H, which defines the new structure of reciprocal space – the so called magnetic Brillouin zone (MBZ). It is a subset of the Brillouin zone, MBZ $\subset$ BZ, i.e. it is rarefied in comparison with the BZ. The maximal abelian subgroup defining MBZ can be chosen in several ways, however, the most popular choice consists in rarefication of quasimomenta along selected axis, say y. The number of elements of MBZ in this direction is q times smaller in comparison with the BZ.

Non-abelian properties of MTG lead to additional rarefaction of the Brillouin zone, this time in the direction perpendicular to the previous one. It is connected with equivalence relation between irreducible representations $\Gamma^{\kappa_x,\kappa_y,s}$ of MTG. The relation between irreducible representations of MTG emerges as a natural consequence of the induction procedure. Along this procedure induction should be carried out for representatives of the orbit of an action of MTG on the set of irreducible representations of the subgroup H. This determines the relations between parameters κ_x and κ_x'. If they fulfill the condition $\kappa_x' = \kappa_x + \eta j \bmod 1$, where $j \in \mathbb{Z}$, then corresponding representations $\Gamma^{\kappa_x,\kappa_y,s}$ and $\Gamma^{\kappa_x',\kappa_y,s}$ are equivalent, what implies the equivalence between k_x and k_x'. As a consequence, the set of non-equivalent quasimomenta are rarefied along x direction.

The Brillouin zone should be defined over non-equivalent quasimomenta. Eventually, the Brillouin zone in magnetic field is q^2-tuply rarefied and results in the magnetic band Brillouin zone (MBBZ), i.e. exactly this set of quasimomenta in the reciprocal space which serves as the support of the energy band structure.

The process of rarefication is connected with the condensation of states, because the total number of states should be conserved. This increases the degeneration of the energy band when going from BZ to MBZ, and eventually to MBBZ. As a result the structure of energy bands consists of q bands, each of them being q-tuply rarefied. The states are labeled by quantum numbers κ_x, κ_y and γ, γ'. Parameters κ_x, κ_y create the MBBZ, whereas the index γ' labels magnetic subbands and γ is associated with the degeneration of an energy level. Introducing the description of the state through the density matrix one can define a measure of the concurrence between states: the first given by $\kappa_x, \kappa_y \in$ MBBZ and index γ' and the second characterized by quasimomentum $\boldsymbol{k} \in$ BZ. This measure allows to describe, which states from BZ overlap with states $\boldsymbol{\kappa} \in$ MBBZ, and in this way it demonstrates quantitatively the effect of a condensation.

H8. A. Wal, *Energy bands for finite two-dimensional systems in a quantised magnetic field: the symmetry of the model*, 2013, Journal of Mathematical Chemistry, **51**, 2285–2316.
The paper concerns energy bands of a two-dimensional lattice in a quantized magnetic field. The key role in the description of such a system plays the symmetry of the model described by the magnetic translation group together with symmetric and unitary groups. The last two groups are used to the analysis of multi-electron states according to duality of Weyl.

The paper presents, in compact form, methods used for the description of an electron in a magnetic field containing in other author's papers. Its important novelty is presentation of results and the method used by the author in previous papers in relation to the achievements of other researchers. This allows to demonstrate research in the wider context and point out possible applications, also in fields other than physics, for example in quantum chemistry. These topics are considered particularly in introduction and during the discussion concerning properties of the magnetic translation group and its representations. In the chapter devoted to multi-electron states, references related to "non-physical" representations are collected and discussed. This extension of work by review of papers related to the subject allows to determine mutual relations between various methods used in the analysis of the problem of Bloch electron in a magnetic field.

The paper starts with the extensive introduction devoted to the problem of an electron in a magnetic field. It contains the review of methods used to solve this problem, starting from free electron model and ending on the analysis of a structure of energy bands, defined over magnetic Brillouin zone for the Bloch electron in a quantized magnetic field. In next chapters, discussion concerns following topics: the structure of the magnetic translation group, mutual relations between gauge and applied boundary conditions, irreducible representations of MTG. The induction procedure for irreducible representations is presented in detail, even for multi-electron systems. The key role in the investigation was played by the group theory. Thanks to the contemporary computer programs, group theory methods, which are considered as tedious and complicated, can be used effectively to solve problems concerning electrons in a periodic potential and a magnetic field.

Scientific publications listed in the data base Web of Science
(state as on 14 August 2014)
Total number of publications: **31**
Total impact factor according to the listing of the Journal Citation Reports (JCR), by the year of publishing: **26,71**
Number of quotations of publications (according to the database Web of Science): **74**
Number of quotations, excluding self-quotations: **43**
Average number of quotations per one publication: **2.39**
The Hirsch index (according to the database Web of Science): **4**

Full listing of publications and conference reports is provided in Enclosure No. 5 (including requirements defined in the regulation of the Minister of Science and Higher Education, dated 1 September 2011, Official Journal No. 196. Item 1165).

5. Description of other scientific and research achievements

a) Description of the scientific career before obtaining the doctorate degree

My scientific interest, after my first employment as an assistant, concerned application of group theory to the description of low-dimensional finite crystals. At that time I focused on determination of solutions of the Heisenberg magnet with the help of Bethe Ansatz. Results of such investigations were published in several papers, in which I analyzed the structure of solution of the Heisenberg magnet, in particular, from the point of view of hidden symmetry defined by the group of authomorphism. The cooperation, concerning this topic, with dr M. Kuzma from Rzeszow and professor T. Lulek from Poznan, resulted in contribution in series of conferences "Symmetry and Structural Properties of Condensed Matter" organized by Adam Mickiewicz University from Poznan, and dedicated to symmetry and structural properties of matter.

Results of research were presented in my doctor's thesis "Crystallographic and magnetic gauge symmetries of a finite linear chain". The promoter was prof. T. Lulek from Adam Mickiewicz University in Poznan, where the public discussion on my thesis was conducted. In my work I considered the symmetry of finite systems according to the Weyl's recipe, which suggests investigation of the hidden symmetry related to the group of automorphisms of the group of obvious symmetry of the system. By analogy to space groups, the whole symmetry of the system is described by an extension of the group of automorphism group by the symmetry group of the model, i.e. translational group. These extensions for linear finite chains were classified in the base of Mac Lane's method. The second part of the thesis was devoted to properties of energy bands of the Heisenberg linear magnet with spins $s = 1/2$ on each node. The problem was analyzed with the help of translational and permutation symmetry. It allowed to determine finite analogue of energy band. The existence of rarefied bands, i.e. bands defined only over selected quasimomenta k from the Brillouin zone (BZ) was pointed out. Such an approach to analysis of energy spectrum gave opportunity to investigate the influence of number of nodes on the shape of energy bands.

During that time I was involved also in the research concerning the influence of laser radiation on distribution of mercury in the semiconductor $Cd_xHg_{1-x}Te$. I constructed the algorithm and the computer program for the simulation of space-time distribution of temperature and concentration of mercury in that particular semiconductor.

b) Description of the scientific career after having obtained the doctorate degree

After PhD, my scientific interest was focused on two topics. I continued the research concentrated on the one-dimensional Heisenberg magnet. Together with the colleagues from Rzeszow, Poznan and professor Caspers from University of Twente (Enschede, Netherlands) we applied method, developed by him, to find the solution of the eigenvalue problem for the finite Heisenberg chain. This approach is based on the asymptotic solutions, which are easy to obtain for large number of nodes of the Heisenberg chain. These asymptotic results are treated as a starting point for determination of consecutive solutions for decreasing number N of nodes. It is assumed, that the change of obtained results, during the decreasing of length of the chain, is quasi-continuous for fixed values of parameters of Bethe equations, i.e. winding numbers λ_i.

However, for the special value N_S of length, this continuity is broken and one can observe a change of character of solutions, for example the real solution transforms into the complex one. Depending on the kind of solutions, on both sides of N_S, we distinguished three types of special points: critical points, transition points and limiting points.

At that time I was involved in investigations of the structure of energy bands of the Heisenberg chain. Thanks to the application of Bethe Ansatz and asymptotic method mentioned above, we analyzed the structure of energy bands for the finite and small number N as a function of quasimomentum and the total spin. We distinguished, in the spectrum, bounded and scattered states. The lowest energy band of ferromagnetic states was described with the help of the rotational band model[17]. For a small system, $N < 10$, we observed small difference between results obtained by the rotational model and Bethe equations.

Determination of solutions of the eigenvalue problem of the Heisenberg magnet XXX with spin $s = 1/2$ can be improved by application of the basis of wavelets. It allows to decrease (N times) the dimension of a secular problem by using the translational symmetry described by the cyclic group C_N. Classical configuration space $Q^{(r)}$ is reduced then to subspaces spanned on the orbits of the group C_N, where r means the number of spin deviations, i.e. the number of spins with the projection opposite to the vacuum state (in the vacuum state projections of spins are parallel). Within this method the matrix of Hamiltonian defined in the space $Q^{(r)}$ reduces, for each quasimomentum k from the Brillouin zone, to submatrices defined in configuration subspace $T = Q^{(r)}/C_N$.

The main result of the research concerning this topic was a full characterization of solutions of the finite one-dimensional Heisenberg magnet, also with the use of the combinatorial object called "rigged string". Within the method, it is possible to find and classify, on the combinatorial way, eigenvalues without the need of solving secular equations.

My second topic of research, after PhD, was investigations of a two-dimensional finite system subjected to a quantized magnetic field. This research became dominant in last six years and resulted in a series of papers devoted to application of the magnetic translation group to the description of an electron in a periodic potential and quantized magnetic field. This topic is discussed in details in the part containing description of scientific achievement.

[17] J. Schnack, M. Luban, Phys. Rev. B **63,** 014418 (2000).

c) Scientific and research plans for the nearest years

In the nearest future I plane to combine, till now conducted separately, topics concerning the one-dimensional Heisenberg chain and two-dimensional lattices in a magnetic field. It can be achieved by the use of the method of algebraic Bethe Ansatz (BA). The eigenvalue problem for selected quasimomenta from the Brillouin zone of an electron in a two-dimensional periodic potential and quantized magnetic field can be solved with the use of BA [18]. Within this context, particularly interesting seems to be the use of the string hypothesis, which allows to find asymptotic solutions for a large value of q, the parameter describing a magnetic field.

On the base of the results already obtained the promising, but difficult topic, is determination of physically correct and mathematically treated boundary conditions for nanoscopic systems subjected to a magnetic field. The new scientific center built at the Faculty of Mathematics and Natural Sciences of University of Rzeszów gives the opportunity to combine the mathematical intuition with the nanotechnology. This can be possible with the help of the scientific equipment, which allows, in particular, production of low-dimensional structures using MBE method and nanolithography. There are also instruments for measuring electron transport in such structures at very low temperatures and strong magnetic field.

d) Distinctions as a result of scientific research

1997	II degree award of the Rector of the Pedagogical University in Rzeszow, 1994, for scientific accomplishments.
2008, 2010, 2012	Awards of the Dean of the Faculty of Mathematics and Natural Science of the University of Rzeszow, three awards, for accomplishments in the scientific work.

e) Membership in organizations/ Functions performed

Since 2003	Council of the Faculty of Mathematics and Natural Science of the University of Rzeszow, member
Since 2004	Rzeszow Branch of the Polish Physical Society, member
Since 2010	Subcarpathian Renewable Energy Cluster, Ecoenergetics Carpathia Region Association, member

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[18] P. Wiegmann and A. Zabrodin, Phys. Rev. Lett. **72**, 1890–1893 (1994); K. Hoshi, and Y. Hatsugai, Phys. Rev. B **61**, 4409–4412 (2000).

f) Professional experience in national and international scientific or academic centers

1995	Bayreuth University, Bayreuth, Germany, DAAD fellowship, 5 months
2003, 2005	University of Twente, Enschede, Nederlands, short visits (one week)
2009	University of Silesia, Katowice, Poland, postdoc., 4 months
2012	University of South Australia, Adelaide, Australia, short visit (10 days)

Rzeszów, 14th August 2014

Andrzej Wal