Summary of Scientific Accomplishments

1 Name

Adam Sawicki

2 Academic degrees

23.09.2014 - PhD in Mathematics, University of Bristol, UK.

Dissertations: : Topology of graph configuration spaces and quantum statistics.

Advisors: Prof. Jonathan P. Keating, Prof. Jonathan M. Robbins.

14.11.2011 - PhD in Theoretical Physics, University of Warsaw, Warsaw.

Dissertaitions: Symplectic Geometry of Entanglement.

Advisor: Prof. dr hab. Marek Kuś.

26.02.2010 - MSc in Theoretical Physics, University of Warsaw, Warsaw.

Dissertations: Classical nonintegrability of a quantum chaotic $SU(3)$ Hamiltonian system.

Advisors: Prof. dr hab. Marek Kuś, Prof. dr hab. Iwo Białynicki-Birula.

3 Academic employment

1. 10/2016-.... Assistant Professor, **Centre for Theoretical Physics Polish Academy of Sciences**, Warsaw.
2. 10/2015-10/2016 - Marie Curie Research Fellow, **School of Mathematics, University of Bristol**, Bristol, UK.
3. 10/2013-10/2015 - Marie Curie Research Fellow, **Centre for Theoretical Physics, Massachusetts Institute of Technology**, Cambridge, USA.
4. 10/2013-10/2016 - Adiunkt, **Centre for Theoretical Physics, Polish Academy of Sciences**, Warsaw (on leave).

AS

5. 10/2010-10/2013 - PhD student, **School of Mathematics, University of Bristol**, Bristol, UK.
6. 03/2010-10/2013 - Assistant, **Centre for Theoretical Physics, Polish Academy of Sciences**, Warsaw (10/2010-10/2013 on leave).
7. 03/2008-03/2010 - Trainee Researcher, **Centre for Theoretical Physics, Polish Academy of Sciences**, Warsaw.

4 The scientific accomplishment

Title of the scientific accomplishment

The momentum map and quantum correlations

(b) Publications for the scientific accomplishment

1. Maciążek T., Sawicki A, 2015 Critical points of the linear entropy for pure L-qubit states J. Phys. A: Math. Theor. 48 045305
2. Oszmaniec M., Suwara P., Sawicki A., 2014, Geometry and topology of CC and CQ states, J. Math. Phys. 55, 06220
3. Sawicki A., Oszmaniec M., Kuś M., 2014, Convexity of momentum map, Morse index, and quantum entanglement, Reviews in Mathematical Physics, 26, 1450004
4. Maciążek T., Oszmaniec M., Sawicki A., 2013, How many invariant polynomials are needed to decide local unitary equivalence of qubit states?, J. Math. Phys. 54, 092201
5. Sawicki A., Tsanov V. V., 2013, A link between quantum entanglement, secant varieties and sphericity, J. Phys. A: Math. Theor. 46 265301
6. Huckleberry A., Kuś M., Sawicki A., 2013, Bipartite entanglement, spherical actions and geometry of local unitary orbits, J. Math. Phys. 54, 022202

(c) Summary of the results of the scientific accomplishment

The structure of the summary is as follows: In section 4.1 we introduce the subject of the research contained in publications [1-6] and present a short description of the main results. In section 4.2 we discuss, both in general and quantum correlations contexts, the key mathematical object, i.e. the momentum map. This section is a starting point for considerations described in sections 4.3, 4.4 and 4.5 that present more detailed discussion of the results obtained in [1-6]. Finally, the short review of other scientific accomplishments is given in section 5.

AJ

4.1 Introduction

A typical quantum system has an infinitely dimensional Hilbert space. An example might be a harmonic oscillator or a hydrogen atom whose Hamiltonians have infinite number of energy levels. Nevertheless, in many cases such systems are treated as finite-dimensional as for example only a few energy levels are important when a system is coupled with the electromagnetic field with a fixed frequency spectrum. The Hilbert space of such finite-dimensional system will be denoted by $\mathcal{H}$. Despite considerable interest in recent years, understanding of correlations in multipartite finite dimensional quantum systems is still incomplete. Natural methods of analysis for such systems are methods of linear algebra. They allow to answer many interesting questions concerning wide range of properties of quantum correlations [55, 33]. They, however, do not give a deeper insight into the geometric structure of the space of states. The existence of this structure enables the use of recently developed advanced methods of complex algebraic/symplectic geometry. Publications [1-6] discussed here are the result of a research programme whose goal was to fill this gap and apply these advanced methods in the context of quantum correlations.

One of the basic problems in the theory of quantum correlations is the classification of states with respect to local operations performed independently on subsystems of a given system. There are two main classes of such operations: (1) local unitary - hereinafter referred to as LU, (2) SLOCC - Stochastic Local Operations with Classical Communication. Mathematically, these operations correspond to the action of some compact group $K \subset SU(\mathcal{H})$ in case (1) and its complexification $G = K^{\mathbb{C}}$ in case (2) on the space of states. The space of pure states (after neglecting the global phase) is the projective space $\mathbb{P}(\mathcal{H})$, and for mixed states, the space of isospectral density matrices is the adjoint orbit of the unitary group $SU(\mathcal{H})$ action. The key fact is that in both cases, these spaces have a natural geometric (Kähler) structure, and therefore in particular they are symplectic manifolds. Moreover, the action of the compact group on M preserves the symplectic structure which yields the existence of the momentum map. This map, in the cases considered here, assigns to a state of L particles its reduced one-particle density matrices, and therefore is directly related to the partial trace over $L-1$ particles. This identification opens up new possibilities as it provides a well-developed tools and methods of algebraic/symplectic geometry in the context of one-particle density matrices. Using this apparatus in the series of paper [1-6], published together with my colleagues:

- I show when information contained in one-particle density matrices is sufficient to solve the problem of LU-equivalence and provide full characterisation of geometric structure of sets of LU-equivalent states [6].
- I show that the existence of the so-called exceptional states is an obstacle for solving the LU-equivalence problem using only one-particle reduced density matrix [5]
- For the many qubit system I show how many additional K-invariant polynomials (except those directly derived from one-qubit density matrices) are needed to solve the LU-equivalence problem. This number varies depending on the spectra of the reduced matrices and I describe how [4].

AS

- I propose a new, more coarse, classification of states under SLOCC operations. It always gives a finite number of generalised SLOCC classes. The proposed method covers pure states of distinguishable particles as well as fermions and bosons [3].

- I present an algorithm for finding critical points of the linear entropy for any number of qubits [1]. This algorithm significantly improves the method proposed in [3].

- I give the geometric and topological characterisations of two-particle mixed states with zero discord, more specifically of CC and CQ states. Sets of these states are closure of the set of symplectic orbits of $SU(N_1) \times SU(N_2)$ and $SU(N_1) \times I_{N_2}$, where N_1 and N_2 are the dimensions of Hilbert spaces of both particles. In addition, I show that these are the only states for which orbits of the considered groups have nonvanishing Euler-Poincareé characteristics [2].

4.2 The momentum map

4.2.1 General setting

In this section we introduce and discuss the momentum map in general setting [24]. Then we give its interpretation in a quantum correlations setting. As we mentioned in the introduction, the momentum map appears always when a Lie group acts on a symplectic manifold preserving the symplectic structure. In the following K will be always a connected compact semisimple matrix Lie group acting in a smooth and symplectic way on a symplectic manifold (M,ω). For each element of the Lie algebra $\xi \in \mathfrak{k} = Lie(K)$ of the group K the fundamental vector field $\hat{\xi} \in \chi(M)$ is assigned in the standard way. The map $\hat{}\colon (\mathfrak{k}, [\cdot,\cdot]) \to (\chi(M), [\cdot,\cdot])$ is a homomorphism of Lie algebras. As M is a symplectic manifold the fundamental vector fields are Hamiltonian. Therefore if the manifold M has the trivial first de Rham cohomology group then for the vector field $\hat{\xi}$ there is a well defined function μ_ξ, such that $d\mu_\xi = \omega(\hat{\xi},\cdot)$. The functions μ_ξ can be chosen to be linear in $\xi \in \mathfrak{k}$, and if the group K is semisimple the mapping $\xi \to \mu_\xi$ is a Lie algebras homomorphism from $\mathfrak{k}$ to $\left(C^\infty(M), \{\cdot,\cdot\}\right)$, where $\{\cdot,\cdot\}$ is the standard Poisson bracket induced by the symplectic form ω. Under these assumptions, we also obtain the unique map $\mu : M \to \mathfrak{k}^*$ defined by $\langle \mu(x), \xi \rangle = \mu_\xi(x)$ that is called *the momentum map*.

Note that the group K acts also on its Lie algebra $\mathfrak{k}$ by the adjoint action $\mathrm{Ad}_g\xi = g\xi g^{-1}$. The dual to this action is the coadjoint action of K on $\mathfrak{k}^*$. For a semisimple K the momentum map is equivariant, i.e. $\mu\left(\Phi_g(x)\right) = \mathrm{Ad}_g^*\mu(x)$ for any $x \in M$ and $g \in K$. Orbits of K-action on M are therefore mapped (by μ) onto orbits of the coadjoint action in $\mathfrak{k}^*$. Coadjoint orbits are in turn symplectic manifolds equipped with the canonical symplectic structure - the so-called Kirillov-Kostant-Souriau form. Moreover, for compact K, coadjoint orbits can be identified with adjoint orbits by means of K-invariant scalar product on $\mathfrak{k}$. Therefore in the following we will always treat μ as the map from M to $\mathfrak{k}$ rather than $\mathfrak{k}^*$.

4.2.2 The momentum map in many particle quantum systems

In publications [1-6] we consider the following spaces M of states:

1. The complex projective space $\mathbb{P}(\mathcal{H})$, where:

AΛ

(a) $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_L$, $\mathcal{H}_i$ is an N_i-dimensional Hilbert space of i-th particle - pure states of L distiguishable particles with a natural actions of $K = SU(N_1) \times \ldots \times SU(N_L)$, $G = K^{\mathbb{C}} = SL(N_1, \mathbb{C}) \times \ldots \times SL(N_L, \mathbb{C})$ and Lie algebras $\mathfrak{k} = \mathfrak{su}(N_1) \oplus \ldots \oplus \mathfrak{su}(N_L)$, $\mathfrak{g} = \mathfrak{sl}(N_1, \mathbb{C}) \oplus \ldots \oplus \mathfrak{sl}(N_L, \mathbb{C})$.

(b) $\mathcal{H} = S^L\mathcal{H}_1$, $\mathcal{H}_1$ is a Hilbert space of a single boson - pure states of L d-state bosons with the diagonal actions of $K = SU(d)$, $G = K^{\mathbb{C}} = SL(d, \mathbb{C})$ and Lie algebras $\mathfrak{k} = \mathfrak{su}(d)$, $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$.

(c) $\mathcal{H} = \bigwedge^L \mathcal{H}_1$, $\mathcal{H}_1$ is a Hilbert space of a single fermion - pure states of L d-state fermions with the diagonal actions of $K = SU(d)$, $G = K^{\mathbb{C}} = SL(d, \mathbb{C})$ and Lie algebras $\mathfrak{k} = \mathfrak{su}(d)$, $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$.

2. Isospectral denisty matrices $\mathcal{O}_\rho$ for the system of two distinguishable particles described by Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, with the adjoint action of $K = SU(N_1) \times SU(N_2)$ and Lie algebra $\mathfrak{k} = \mathfrak{su}(N_1) \oplus \mathfrak{su}(N_2)$.

All the above given spaces are Kähler manifolds so in particular they are symplectic manifolds. The formula for the symplectic form at a point $[v] \in \mathbb{P}(\mathcal{H})$ or $\sigma \in \mathcal{O}_\rho$ is given by:

$$\omega(\hat{\xi_1}, \hat{\xi_2}) = \frac{-i\langle v|[\xi_1, \xi_2]v\rangle}{2\langle v|v\rangle}, \quad \omega(\hat{\xi_1}, \hat{\xi_2}) = \frac{-i}{2}\mathrm{Tr}(\sigma[\xi_1, \xi_2]),\ \xi_i \in \mathfrak{k}, \tag{1}$$

respectively. The momentum map $\mu : \mathbb{P}(\mathcal{H}) \to \mathfrak{k}$ is given by:

1. Pure states of L distinguishable particles:

$$\mu([v]) = \frac{i}{2}\{\rho_1([v]) - \frac{1}{N_1}I_{N_1}, \rho_2([v]) - \frac{1}{N_2}I_{N_2}, \ldots, \rho_L([v]) - \frac{1}{N_L}I_{N_L}\}, \tag{2}$$

where $\rho_i([v])$ is the i-th reduced one-particle density matrix of a state $[v] \in \mathbb{P}(\mathcal{H})$ and I_{N_i} is the identity operator on the N_i-dimensional Hilbert space $\mathcal{H}_i$.

2. Pure states of L d-state bosons or fermions: $\mu([v]) = \frac{i}{2}(\rho_1([v]) - \frac{1}{d}I)$, $\rho_1([v])$ is one-boson/one-fermion reduced density matrix.

4.3 K-equivalence of quantum states

Publications [4,5,6] concern the problem of the local unitary equivalence of pure quantum states. Two states are called local unitary equivalent (K-equivalent) if they can be connected by K-action. The aim of the work [4,5,6] was to investigate when the K-equivalence can be solved using only the momentum map and for cases when it is not possible to identify a physical reason of this inability and check how much information is missing. Note that in all the considered systems the momentum map is directly related to the operation of partial trace and therefore to 1-particle reduced density matrices. We can thus say that publications [4,5,6] examine the role of 1-particle reduced density matrices in the problem of K-equivalence.

Note that since K is a compact group the K-orbits are closed. In order to check if two states belong to the same K-orbit it suffices to find (the finitely generated [52]) ring of

AO

K-invariant polynomials, i.e. polynomials that are constant on K-orbits. Let us recall that the group K also acts on its Lie algebra through the adjoint action. The existence of the equivariant momentum map $\mu : M \to \mathfrak{k}$ guarantees that K-orbits in the space of states M are mapped by μ onto adjoint orbits in $\mathfrak{k}$. Therefore, if $p : \mathfrak{k} \to \mathbb{R}$ is an Ad_K-invariant polynomial on $\mathfrak{k}$ then the composition $p \circ \mu : M \to \mathbb{R}$ is a K-invariant polynomial on M. $\mathrm{Ad}_{\mathfrak{su}(N)}$-invariant polynomials are known. They are given by $\{\mathrm{Tr}X^2, \mathrm{Tr}X^3, \ldots, \mathrm{Tr}X^N\}$, where $X \in \mathfrak{su}(N)$. The Lie algebras considered in [4-6] are either the special unitary Lie algebra $\mathfrak{su}(d)$ (for d-level bosons and fermions) or the direct sum of special unitary Lie algebras $\mathfrak{k} = \mathfrak{su}(N_1) \oplus \ldots \oplus \mathfrak{su}(N_L)$ (for L distinguishable particles). Combining this with the fact that the momentum map is given by reduced one-particle density matrices we get that traces of their powers are K-invariant polynomials on M. If the momentum map has a property that the pre-image of every adjoint orbit from $\mu(M) \subset \mathfrak{k}$ is exactly one K-orbit in M, then the K-invariant polynomials on M and Ad_K-invariant polynomials on $\mathfrak{k}$ are in 1-1 correspondence (given by μ). Let us note that for $X \in \mathfrak{su}(N)$ the values of $\mathrm{Ad}_{\mathfrak{su}(N)}$-invariant polynomials at X determine the spectrum of X (coefficients of the characteristic polynomial for X are expressible in terms of traces of powers of X). Therefore the K-equivalence of two states in M can be checked in this case by comparing spectra of their reduced one-particle density matrices. Typically this is not the case and many K-orbits are mapped onto one adjoint orbit in $\mu(M) \subset \mathfrak{k}$. As a direct consequence, for two states whose reduced one-particle density matrices have the same spectra we need additional K-invariant polynomials to decide their K-equivalence. This kind of polynomials were found for some low-dimmesional systems [16, 45, 66, 69]. Thus in [4-6] we focus not on finding K-invariant polynomials that typically have no well defined physical meaning, but on the geometric properties of the considered problem. In the following sections we describe problems solved in [4-6].

4.3.1 The role of sphericity in K-equivalence

As we already pointed out in the previous section, the situation when the set of K-invariant polynomials on M is given by the composition of Ad_K-invariant polynomials on $\mathfrak{k}$ with the momentum map $\mu : M \to \mathfrak{k}$ occurs only when the pre-image of every adjoint orbit from $\mu(M) \subset \mathfrak{k}$ is exactly one K-orbit in M. Let us denote by $\mathcal{F}_x := \mu^{-1}(\mu(x))$ the fibre of the momentum map μ over $\mu(x) \in \mathfrak{k}$. The position of $\mathcal{F}_x$ with respect to the orbit $K.x$ is of the key importance for the above situation. Orbits of K in M are in 1-1 correspondence with the adjoint orbits in $\mu(M)$ if and only if the fibres $\mathcal{F}_x$ are contained in orbits $K.x$. It is easy to see that $T_x\mathcal{F}_x \subset (T_xK.x)^{\perp_\omega}$ which means that if $K.x$ is coisotropic then $\mathcal{F}_x$ is contained in it. In order to characterise all systems for which K-equivalence can be decided using μ we need to identify those whose (at least) generic K-orbit is coisotropic. As it turns out, such systems need to satisfy some group theoretic conditions which we analyse in [6].

Interestingly, it is crucial for the considered problem to study not only the action of K but also its complexification $G = K^{\mathbb{C}}$. Note that since K is a maximal compact subgroup of G, the group G is *reductive*. An important example of the reductive group is $G = \mathrm{SL}_N(\mathbb{C})$ which is complexification of its compact subgroup $K = \mathrm{SU}(N)$. By this example groups G considered in [6] (and introduced in section 4.2.2) are reductive. Note that the group G is much bigger than K and therefore the number of G-orbits is smaller than nuber of K-orbits in M. If G has an open dense orbit $\Omega = G/H$ on M then we call M an almost

AJ

homogenous manifold [34]. In particular such M has a finite number of G-orbits. The preliminary considerations contained in [6] lead to the conclusion, that almost homogeneity of M with respect to G-action is a necessary condition for deciding K-equivalence using the momentum map. It is, however, not a sufficient condition which can be seen on the example of three-qubit system, where there are exactly 6 orbits of $G = SL(2,\mathbb{C})^{\times 3}$ action [20], but states $x_1 = \sqrt{\frac{2}{3}}|000\rangle + \frac{1}{\sqrt{3}}|111\rangle$ and $x_2 = \frac{1}{\sqrt{3}}(|000\rangle + |010\rangle + |001\rangle)$, where $\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2$ is an orthonormal basis in $\mathbb{C}^2$, satisfy $\mu(x_1) = \mu(x_2)$ and are not K-equivalent as they belong to different G-orbits [20].

An important role in the formulation of the sufficient condition is played by the Borel subgrup of the group G. Let us recall that by definition a Borel subgroup B is a maximal connected solvable subgroup of the group G. For example, for $G = \mathrm{SL}_N(\mathbb{C})$, the group of upper-triangular matrices, that is a stabilizer of the standard full flag in $\mathbb{C}^N$:

$$0 \subset \mathrm{Span}\{|1\rangle\} \subset \mathrm{Span}\{|1\rangle, |2\rangle\} \subset \ldots \subset \mathrm{Span}\{|1\rangle, \ldots, |N-1\rangle\} \subset \mathrm{Span}\{|1\rangle, \ldots, |N\rangle\} = \mathcal{H},$$

is an example of a Borel subgroup. Generally, any two Borel subgroups are conjugated by an element of G. Therefore, in the considered example, B is a Borel subgroup of G if and only if it stabilises some standard full flag. The crucial notion for the K-equivalence problem is the notion of a *spherical space.* G-homogenous space $\Omega = G/H$ is a spherical homogenous space if and only if some and therefore every Borel subgroup $B \subset G$ has an open dense orbit in Ω. If G has an open dense orbit $\Omega = G/H$ in M and Ω is a spherical homogenous space then M is the spherical embedding of $\Omega = G/H$. Such M is also called an almost homogenous spherical space. The momentum map separates K-orbits on almost homogenous spherical spaces. This is a conclusion from the Brion theorem [13], which says:

Theorem 1 *(Brion) Let K be a connected compact Lie group acting on a connected compact Kähler manifold (M,ω) by a Hamiltonian action and let $G = K^{\mathbb{C}}$. The following are equivalent*

1. *M is a spherical embedding of the open G-orbit.*
2. *For every $x \in M$ the μ-fiber $\mu^{-1}(\mu(x))$ is contained in $K.x$.*

In [6] we use Brion's theorem and show that the open dense orbit of the Borel subgroup exists only for systems of two fermions, two bososons and two distinguishable particles. Therefore, these are the only systems for which K-equivalence can be decided using reduced one-particle density matrices.

In the second part of [6], for systems satisfying conditions of Brion's theorem, we describe geometric structure of K-orbits. Adjoint orbits are symplectic manifolds. K-orbits in M typically are not symplectic. The reason for this is vanishing of the symplectic form on the tangent space to the momentum map fibers that are contained inside K-orbits. Thus on K-orbits we have only a partial symplectic structure. We can describe this partial structure as by Brion's theorem the momentum map $\mu : M \to \mathfrak{k}$ parametrises K-orbits in M in the sense that it bijectively maps the set of K-orbis in M onto K-orbits in $\mu(M)$. The image is given by $K.P$ where P is a convex subset of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$. In [6] we describe P as a probability polyhedron. We also find a real algebraic set $\Sigma^+_{\mathbb{R}}$ in M, defined by linear algebraic

AD

equations and inequalities, that parameterizes the K-orbits in M and that is mapped onto a fundamental region of the Weyl-group in P by the momentum map. Every element x of $\Sigma_{\mathbb{R}}^{+}$ determines in a simple way a vector $d = (d_1, \ldots, d_k)$ of positive integers that completely determines the momentum map image $\mu(x)$ as a flag manifold $F(d_1, \ldots, d_k)$. We also exactly describe the fiber $\mathcal{F}_x = \mu^{-1}(\mu(x))$ of the momentum map. It is the fiber of the homogeneous fibration $K/K_x \to K/K_{\mu(x)}$ which in fact (up to very simple finite-coverings) is a product of certain symmetric spaces. In the case of bosons it is the product of a torus and a number (depending on d and the degeneracy) of symmetric spaces of the form $\mathrm{SU}_m/\mathrm{SO}_m$. The case of fermions is analogous, except that the symmetric spaces are of the form $\mathrm{SU}_m/\mathrm{USp}_m$.

4.3.2 Exceptional states, sphericity and K-equivalence

In the previous section we gave necessary and sufficient conditions for deciding K-equivalence of pure states using only the momentum map. It was the requirement that the space of states M is a spherical embedding of $G = K^{\mathbb{C}}$-homogenous space. In [5] we show that the existence of the so called exceptional states in M is an obstacle for sphericity of M. As we prove, exceptional states exist in $\mathbb{P}(\mathcal{H})$ for all considered systems of distinguishable and indistinguishable particles except the case when their number is $L = 2$.

In order to define the notion of an exceptional state we first need to say what are the rank and the border rank [46] of a state in M. The rank of a state is defined with respect to Perelomov coherent states [58] $\mathbb{X}$ of K-action on M. For distinguishable particles $\mathbb{X}$ is the image of the Segre map given by Segre : $\mathbb{P}(\mathcal{H}_1) \times \ldots \times \mathbb{P}(\mathcal{H}_L) \longrightarrow \mathbb{P}(\mathcal{H})$, which has the following action $([v_1], \ldots, [v_L]) \longmapsto [v_1 \otimes \ldots \otimes v_L]$. Thus elements $\mathbb{X}$ are states corresponding to simple tensors. For bosons $\mathbb{X}$ is the image of the Veronese map: $\mathrm{Ver}_L : \mathbb{P}(\mathcal{H}_1) \longrightarrow \mathbb{P}(\mathcal{H})$, which acts as $[v] \longmapsto [v^L]$, and for fermions $\mathbb{X}$ is the image of the Plücker map $\mathrm{Pl}_L : Gr(L, \mathcal{H}_1) \longrightarrow \mathbb{P}(\mathcal{H})$, which has the following action $U \longmapsto [u_1 \wedge \ldots \wedge u_L]$, where $u_1, \ldots, u_L$ is a basis in U. The rank of a state is defined as

$$\mathrm{rk}[\psi] = \mathrm{rk}_{\mathbb{X}}[\psi] = \min\{r \in \mathbb{N} : \psi = x_1 + \cdots + x_r \text{ where } [x_j] \in \mathbb{X}\} \,. \tag{3}$$

The set of states of rank r will be denoted by $\mathbb{X}_r = \{[\psi] \in \mathbb{P}(\mathcal{H}) : \mathrm{rk}[\psi] = r\}$. It is easy to see that $\mathbb{X}_r$ is not closed (in Zariski topology) as we have $\mathbb{X} \subset \overline{\mathbb{X}_r}$ and $\mathbb{X} \not\subset \mathbb{X}_r$. The secant variety of rank r, $\sigma_r(\mathbb{X})$ is a variety that contains the closure of all sets of states of rank at most r: $\sigma_r(\mathbb{X}) = \overline{\bigsqcup_{s \leq r} \mathbb{X}_s} \subset \mathbb{P}(\mathcal{H})$ and is a well defined algebraic variety. It turns out that one can have states $x \in \mathbb{P}(\mathcal{H})$ of the certain rank r that can be approximated with an arbitrary precision with states of strictly lower rank. For any state $[\psi] \in \mathbb{P}(\mathcal{H})$ we define its *border rank* with respect to $\mathbb{X}$ as $\underline{\mathrm{rk}}[\psi] = \underline{\mathrm{rk}}_{\mathbb{X}}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \overline{\mathbb{X}_r}\}$. States satisfying $\underline{\mathrm{rk}}[\psi] < \mathrm{rk}[\psi]$ are called *exceptional states*. Exceptional states turn out to be closely connected to sphericity. In [5] we show that for a projective space $\mathbb{P}(\mathcal{H})$, which is a spherical almost homogenous space (not only for many (in)distinguishable particles), there are no exceptional states. More precisely

Theorem 2 *Let $G \to GL(\mathcal{H})$ be an irreducible representation of a reductive complex Lie group G, such that the action of G on $\mathbb{P}(\mathcal{H})$ is spherical. Let $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$ be the closed G-orbit. Then rank and border rank on $\mathbb{P}(\mathcal{H})$ with respect to $\mathbb{X}$ coincide, i.e.*

$$\mathrm{rk}_{\mathbb{X}}[\psi] = \underline{\mathrm{rk}}_{\mathbb{X}}[\psi] \,,$$

AJ

for all $[\psi] \in \mathbb{P}$. In other words, there are no exceptional states in $\mathbb{P}(\mathcal{H})$.

The proof of this theorem is based on the classification of spherical representations of reductive groups given by Knopp [42].

Therefore, in order to show that a lack of exceptional states in $\mathbb{P}(\mathcal{H})$ is equivalent (for considered many particle systems) to sphericity we need to prove that exceptional states exist always when the number of particles L is grater than 2. An example of an exceptional state is the 3-qubit state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. To see this let us consider one-parameter group given by

$$A(a) = \frac{1}{2}\begin{pmatrix} a+a^{-1} & a-a^{-1} \\ a-a^{-1} & a+a^{-1} \end{pmatrix}, \quad a \in \mathbb{C}^{\times},$$

which can be rewritten as

$$A(a) = g_0 A_1(a) g_0^{-1}, \text{ where } A_1(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in GL_2(\mathbb{C}), \; g_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SU(2).$$

As we show in [5] we have the following convergence in $\mathbb{P}(\mathcal{H})$: $A(a)^{\otimes 3}[|000\rangle + |111\rangle] \xrightarrow{a\to 0} g_0^{\otimes 3}[|011\rangle + |101\rangle + |110\rangle]$. The state $|g_0 W\rangle$ and therefore also $|W\rangle$ can be approximated by states of rank 2. But the direct calculations show that $|W\rangle$ has rank 3. Thus $|W\rangle$ is an exceptional state. As we show in [5], the existence of exceptional states for 3 qubits implies their existence for pure states of $L \geq 3$ distinguishable particles. Similarly, we prove (using results contained in [18] and [15]) that their existence for three bosons and three 6-state fermions (here we use [36]) imply their existence for pure states of $L \geq 3$ indistinguishable particles. The main theorem of [5] reads:

Theorem 3 *Suppose that we have one of the following three configurations of a state space $\mathcal{H}$, a complex reductive Lie group G acting irreducibly on $\mathcal{H}$, and a variety of coherent states $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$, which is the unique closed G-orbit in the projective space $\mathbb{P}(\mathcal{H})$.*

(i) $\mathcal{H}_D = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_L$, $G_D = GL(\mathcal{H}_1) \times \ldots \times GL(\mathcal{H}_L)$, $\mathbb{X} = \mathrm{Segre}(\mathbb{P}(\mathcal{H}_1) \times \ldots \times \mathbb{P}(\mathcal{H}_L))$.

(ii) $\mathcal{H}_B = S^L(\mathcal{H}_1)$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \mathrm{Ver_L}(\mathbb{P}(\mathcal{H}_1))$.

(iii) $\mathcal{H}_F = \bigwedge^L \mathcal{H}_1$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \mathrm{Pl}(\mathrm{Gr}(\mathrm{L}, \mathcal{H}_1))$.

Then the action of G on $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. G_D on $\mathbb{P}(\mathcal{H}_D)$) is spherical if and only if there are no exceptional states in $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. $\mathbb{P}(\mathcal{H}_D)$) with respect to $\mathbb{X}$. In other words, sphericity of the representation is equivalent to the property that states of a given rank cannot be approximated by states of lower rank.

Combining this theorem with the results described in the previous section we get the main result of [5] saying that the existence of exceptional states is an obstacle for deciding K-equivalence of states using the momentum map.

4.3.3 Symplectic reduction and K-equivalence

As we have discussed in the previous sections, the two-particle case ($L = 2$) is the only situation, when the space of pure states is a spherical variety, or, equivalently, when there are no exceptional states. For such quantum systems the K-orbits in M are in the one-to-one correspondence with the adjoint orbits in $\mu(M) \subset \mathfrak{k}$. The problem of K-equivalence of

AS

quantum states is then completely solved by considering the image of the momentum map. In other words, it is enough to compare the spectra of the one-particle reduced density matrices, or the values of polynomials $p_i : \mathfrak{k} \to \mathbb{R}$, which are invariant with respect to the adjoint action. Such polynomials are given by the traces of powers of the one-particle density matrices. When $L > 2$, the knowledge of the adjoint-invariant polynomials $\{p_i\}$ is not sufficient for solving the K-equivalence problem. However, the condition $\mu(K.|\psi\rangle) = \mu(K.|\phi\rangle)$ is necessary for the K-equivalence of states $[\phi]$ and $[\psi]$. The momentum image $\mu(M)$ consists of adjoint orbits in $\mathfrak{k}$. Each adjoint orbit intersects the Cartan subalgebra $\mathfrak{t}$ at a finite number of points, that are connected by the action of the Weyl group. Let us denote by $\mathfrak{t}_+ \subset t$ the positive Weyl chamber and let $\Psi : M \to \mathfrak{t}_+$ be the map that satisfies $\Psi(|\phi\rangle) = \mu(K.|\phi\rangle) \cap \mathfrak{t}_+$. In the considered quantum systems, map Ψ assigns to a state ϕ the ordered spectra of the (shifted) one-particle reduced density matrices. By taking the intersection of $\mu(M)$ with the positive Weyl chamber, $\mathfrak{t}_+ \subset \mathfrak{t}$, one obtains the set $\Psi(M) = \mu(M) \cap \mathfrak{t}_+$, that parametrises adjoint orbits in $\mu(M) \subset \mathfrak{k}$ [24]. The celebrated convexity theorem of the momentum map [8, 25, 38] states that $\Psi(M)$ is a convex polytope, which is also referred to as the Kirwan polytope. The necessary condition for states $[\phi_1]$ and $[\phi_2]$ to be K-equivalent, can be therefore formulated as $\Psi([\phi_1]) = \Psi([\phi_2])$. As we show in [4], for L-qubit states that satisfy the aforementioned necessary condition, the number of additional invariant polynomials strongly depends of the spectra of the one-qubit reduced density matrices, i.e. on the point in the polytope $\Psi(M)$. For $\alpha \in \Psi(M)$, the number of additional polynomials is given by the dimension of the reduced space $M_\alpha = \Psi^{-1}(\alpha)/K$. In [4] we analyse $\dim_{\mathbb{R}} \Psi^{-1}(\alpha)/\mathrm{K}$ for an arbitrary $\alpha \in \Psi(M)$.

Inequalities that describe polytope $\Psi(M)$ for a system of L qubits, are known [31]. Denote by $\{p_i, 1-p_i\}$ an increasingly ordered spectrum of the i-th reduced density matrix and by λ_i the shifted spectrum, $\lambda_i = \frac{1}{2} - p_i$. Then, $\Psi(M)$ is given by $0 \leq \lambda_l \leq \frac{1}{2}$ and $\left(\frac{1}{2} - \lambda_l\right) \leq \sum_{j\neq l}\left(\frac{1}{2} - \lambda_j\right)$. Methods that we use in [4] to compute the dimensions of spaces M_α, are different for points α belonging to the interior of $\Psi(M)$ and for points from the boundary of the polytope. For more than two qubits, the polytope is of full dimension, hence a generic K-orbit in the space of states M has the dimension of K [64]. Using the regularity of μ [29, 49] we get that for points α from the interior of the polytope the dimension of the reduced space reads:

$$\begin{aligned}\dim M_\alpha &= \dim(\Psi^{-1}(\alpha)/K) = (\dim\mathbb{P}(\mathcal{H}) - \dim\Psi(\mathcal{H})) - \dim K = \\ &= \left((2^{L+1} - 2) - L\right) - 3L = 2^{L+1} - 4L - 2. \end{aligned} \tag{4}$$

Points belonging to the boundary of $\Psi(M)$ can be grouped into three classes: (i) k of λ_l are equal to $\frac{1}{2}$, (ii) at least one of inequalities $\left(\frac{1}{2} - \lambda_l\right) \leq \sum_{j\neq l}\left(\frac{1}{2} - \lambda_j\right)$ is an equality, (iii) k of λ_l are equal to 0. In case (i), inequalities that yield $\Psi(M)$ reduce to an analogical set of inequalities for the $(L-k)$-qubit polytope. Therefore, $\dim M_\alpha = \left((2^{L-k+1} - 2) - (L-k)\right) - 3(L-k) = 2^{L-k+1} - 4(L-k) - 2$. States that are mapped to points that fall into case (ii) belong to the $K^{\mathbb{C}}$-orbit through the L-qubit W-state, $[W] = |01\ldots1\rangle + |101\ldots1\rangle + \ldots + |1\ldots10\rangle$ [4]. As we showed in [64], the closure of such an orbit is an almost homogeneous spherical variety. Therefore, fibers of the momentum map are contained in K-orbits, i.e. $\dim M_\alpha = 0$. Case (iii), where k of λ_l are equal to 0, is the most difficult one, as it requires the use of some more advanced tools from the Geometric Invariant Theory (GIT) [52]. In the GIT-theory a key role is played by *stable states* [52, 53], i.e. states for which $\mu([\phi]) = 0$ and $\dim K.|\phi\rangle = \dim K$

AD

[39]. For a symplectic action of a compact group $\tilde{K}$ the existence of stable states implies that $\tilde{\mu}^{-1}(0)/\tilde{K} = \dim\mathbb{P}(\mathcal{H}) - 2\dim\tilde{K}$, where $\tilde{\mu}$ is the momentum map for the $\tilde{K}$-action. The strategy for case (iii) is the following. As we point out in paper [4], group K can be divided into $K = K_1 \times K_2$, where $K_1 = SU(2)^{\times k}$, $K_2 = SU(2)^{\times(L-k)}$ and K_1 acts on the first k qubits. The symplectic action of K_1 yields the momentum map, which assigns to a state its first k one-qubit reduced density matrices. Therefore, $\mu_1^{-1}(0)$ consists of states whose first k reduced density matrices are maximally mixed, while the remaining $(L-k)$ reduced matrices are arbitrary. In paper [4] we also construct a state that is GIT stable with respect to the action of $K_1^{\mathbb{C}}$ on $\mathbb{P}(\mathcal{H})$. Hence, $\dim\mu_1^{-1}(0)/K_1 = \dim\mathbb{P}(\mathcal{H}) - 2\dim K_1 = 2^{L+1} - 6k - 2$. Furthermore, the quotient $\mu_1^{-1}(0)/K_1$ is a symplectic variety itself. Because the actions of K_1 and K_2 commute, we can consider the action of K_2 on $\mu_1^{-1}(0)/K_1$. The momentum map for K_2 acting on $\mu_1^{-1}(0)/K_1$ gives the remaining $L-k$ one-qubit reduced density matrices. The polytope of Ψ_2 is of full dimension, i.e. of dimension $L-k$. By the formula for the dimension of the reduced space for points from the interior of the polytope, we get:

$$\begin{aligned}\big((\dim\mu_1^{-1}(0)/K_1) - (L-k)\big) - \dim K_2 &= \big((2^{L+1} - 6k - 2) - (L-k)\big) - 3(L-k)\\ &= 2^{L+1} - 4L - 2k - 2,\end{aligned}$$

which is the desired result for case (iii).

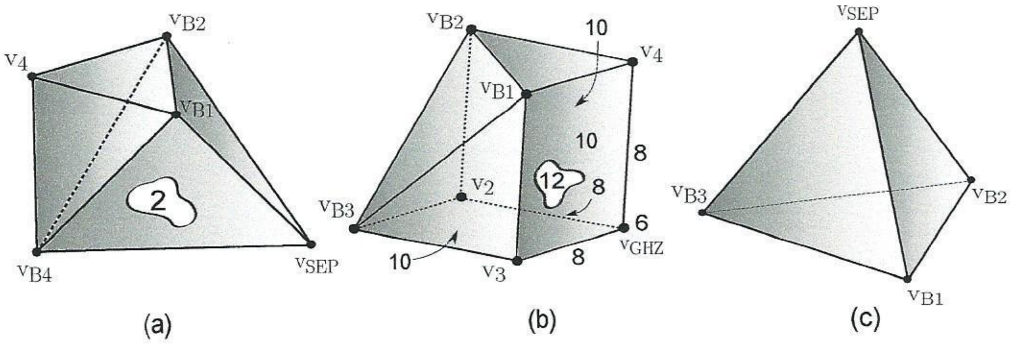

Figure 1: The three parts of the boundary of $\Psi(\mathbb{P}(\mathcal{H}))$ for four qubits. The numbers denote $\dim M_\alpha$. If the number is missing, then $\dim M_\alpha = 0$.

4.4 $K^{\mathbb{C}}$-equivalence of quantum states

Another fundamental problem in the theory of quantum correlations is the classification of states under SLOCC (Stochastic Local Operations and Classical Communication) operations [68]. This classification is still not fully understood. For the considered multipartite systems reversible SLOCC operations correspond to elements of the complexification $G = K^{\mathbb{C}}$ of the local unitary operations group K [41], and two states are G-equivalent if and only if they belong to the same G-orbit. Recall that the problem of K-equivalence is solvable by means of K-invariant polynomials. As the group G is reductive, the Hilbert and Nagata theorem [52] ensures that the ring of G-invariant polynomials is finitely generated. However, the problem of G-equivalence turns out to be significantly different from the problem of K-equivalence. As we note in [3] the essence of this difference is the fact that the group G

is not a compact group, and thus G-orbits do not have to be closed. For two vectors ϕ_1 and ϕ_2 satisfying $G.\phi_1 \cap G.\phi_2 = \emptyset$ we can have $\overline{G.\phi_1} \cap \overline{G.\phi_2} \neq \emptyset$. Note that G-invariant polynomials are continuous functions and therefore they are not able to distinguish between orbits $G.\phi_1$ and $G.\phi_2$. It is only possible to distinguish between orbits whose closures have non-empty intersection, in particular between closed G-orbits. The orbit space M/G, i.e. the quotient space resulting from dividing the state space by the action of the group G, is not a Hausdorff space - not every pair of points is separated by open sets. Therefore, the problem of the G-equivalence of states requires, above all, understanding the structure of the orbit space resulting from the action of a non-compact reductive group on a vector space $\mathcal{H}$ (equivalently on the projective space $\mathbb{P}(\mathcal{H})$).

Two orbits of $G.\phi$ and $G.\psi$ in $\mathcal{H}$ are called c-equivalent[1] iff there exists a sequence of orbits $G.\phi = G.v_1, G.v_2, \ldots, G.v_n = G.\psi$ such that $\overline{G.v_k} \cap \overline{G.v_{k+1}} \neq \emptyset$. The relation of c-equivalence divides G-orbits into equivalence classes (c-classes). It turns out, that in fact, every c-class contains exactly one closed G-orbit, which is contained in the closure of every G-orbit belonging to the considered c-class. The dimension of this orbit is the smallest in the whole c-class. C-equivalence classes are therefore parameterised by closed G-orbits and G-invariant polynomials distinguish between G-orbits belonging to different c-classes [52]. In the next paragraphs we focus on the construction of the quotient space with respect to the c-equivalence relation.

Among all c-classes we distinguish those corresponding to the zero vector - they form the so-called null cone [54]. This class must be removed if we want to consider the quotient space at the projective level. After removing from the projective space $\mathbb{P}(\mathcal{H})$ points corresponding to vectors from the null cone we are left with *semistable* points $\mathbb{P}(\mathcal{H})_{ss}$. Two points $x_1, x_2 \in \mathbb{P}(\mathcal{H})_{ss}$ are c-equivalent if there are vectors of $v_1, v_2 \in \mathcal{H}$, such that $x_1 = [v_1]$ and $x_2 = [v_2]$ and on the level of the Hilbert space $\overline{G.v_1} \cap \overline{G.v_2} \neq \emptyset$. The quotient space obtained from the semistable points by c-equivalence relation is denoted by $\mathbb{P}(\mathcal{H})_{ss} /\!/ G$ and is a projective algebraic variety. It is known in the literature under the name *GIT quotient*[2] [53] and we will call it *GIT space*. Points of the GIT space correspond to c-classes of semistable points and are in one-to-one correspondence with closed G-orbits. So far, we have not used the momentum map. It turns out that every closed G-orbit in $\mathbb{P}(\mathcal{H})_{ss}$ contains exactly one K-orbit from $\mu^{-1}(0)$ [37]. Therefore we get the following equivalence $\mu^{-1}(0)/K \cong \mathbb{P}(\mathcal{H})_{ss} /\!/ G$.

The set of closed G-orbits is given by the action of G on $\mu^{-1}(0)$, i.e. $G.\mu^{-1}(0)$. Among the semistable points we distinguish the so-called stable points $\mathbb{P}(\mathcal{H})_s = \{x \in \mathbb{P}(\mathcal{H})_{ss} : \dim G.x = \dim G \text{ and } G.x \cap \mu^{-1}(0) \neq \emptyset\}$. The existence of a single stable point makes $\mathbb{P}(\mathcal{H})_s$ an open dense subset of $\mathbb{P}(\mathcal{H})_{ss}$, i.e. almost every semistable point is stable [53, 50]. For the stable point $x \in \mathbb{P}(\mathcal{H})_s$ the c-equivalnce class consists of exactly one closed G-orbit. For semistable but not stable points this class always consists of an infinite number of G-orbits.

Vectors belonging to the null cone, i.e. c-class whose closed G-orbit is the zero vector, may represent important states from the point of view of quantum correlations. For example, states $|W\rangle = 1/\sqrt{3}(|001\rangle + |010 + |100\rangle\rangle)$ and separable states belong to the null cone but their quantum properties are significantly different. Therefore we need a finer procedure dividing G-orbits, one that includes the GIT construction and also provides mathematically

[1] From closure equivalent.

[2] Geometric Invariant Theory

AJ

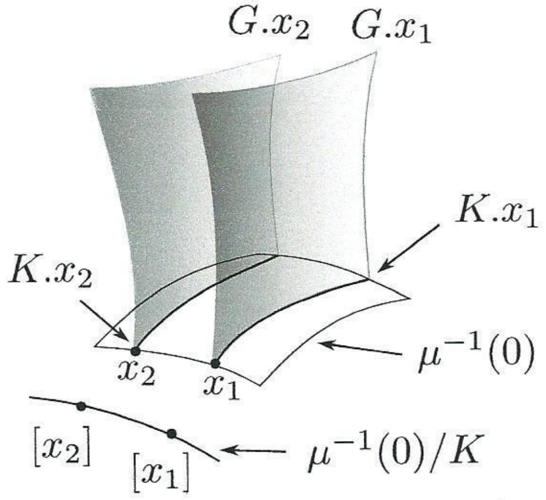

Figure 2: The idea of the categorical quotient construction, i.e. $\mu^{-1}(0)/K \cong \mathbb{P}(\mathcal{H})_{ss} /\!/ G$.

and physically well-defined stratification of the null cone. A key role is played here by the function $||\mu||^2 : \mathbb{P}(\mathcal{H}) \to \mathbb{R}$. This function has a clear physical interpretation. According to the definition of Klyachko [41] the total variance of state $[v] \in \mathbb{P}(\mathcal{H})$ with respect to the symmetry group $K \subset SU(\mathcal{H})$ is given by

$$\mathrm{Var}([v]) = \frac{1}{\langle v|v\rangle}\left(\sum_{i=1}^{\dim K} \langle v|\xi_i{}^2|v\rangle - \frac{1}{\langle v|v\rangle}\sum_{i=1}^{\dim K} \langle v|\xi_i|v\rangle^2\right) = c - 4\cdot ||\mu||^2([v]), \tag{5}$$

where ξ_i is an orthonormal basis of algebra $\mathfrak{k}$ and c is a $[v]$-independent constant. The function $||\mu||^2([v])$ can be also expressed as the expectation value of the Casimir operator $\mathcal{C}_2 = \sum_{i=1}^{\dim K} \xi_i^2$ [11, 27] which acts on $\mathrm{Sym}^2\mathcal{H}$ [57]. In this case we have $\mathcal{C}_2^\vee = \sum_{i=1}^{\dim K}(\xi_i \otimes I + I \otimes \xi_i)^2$ and $\frac{1}{\langle v|v\rangle^2}\langle v\otimes v|\mathcal{C}_2^\vee|v\otimes v\rangle = 2c + 8\,||\mu||^2([v])$. Finally, $||\mu||^2$ is directly related to the linear entropy which is a linear function of the total variance.

A point $[v] \in \mathbb{P}([v])$ is a critical point of $||\mu||^2$ if it is a solution of the eigenproblem $\mu([v]).v = \lambda v$ [3]. Critical points of $||\mu||^2$ can be therefore divided into two categories. The first includes all K-orbits belonging to $\mu^{-1}(0)$. These are called minimal critical points and for them $||\mu||^2$ reaches a global minimum. The minimal critical points correspond to states with maximum total variance and maximum linear entropy. The other critical points are given by some K-orbits in the null cone. For these points $\mu([v]) \neq 0$ and $\mu([v])v = \lambda v$. Therefore, in the null cone we distinguish G-orbits passing through the critical K-orbits.

The relationship between critical points of $||\mu||^2$, c-equivalence and GIT construction becomes clear if we consider the gradient flow of $-||\mu||^2$ [54]. The gradient of $-||\mu||^2$ is well defined as the projective space $\mathbb{P}(\mathcal{H})$ is a Kähler manifold, and therefore in particular have a well defined metric. The gradient flow is tangent to G-orbits and carries points towards critical K-orbits. Two points $x_1, x_2 \in \mathbb{P}(\mathcal{H})_{ss}$ are equivalent from the point of view of the gradient flow if they are taken by it to the same critical K-orbit. This definition is consistent with the c-equivalence definition. However, it is at the same time more general because it allows an extension of the concept of equivalence to the null cone. The situation in the null cone is more complex as the critical K-orbits not need to be in one fiber of Ψ (recall that $\Psi : \mathbb{P}(\mathcal{H}) \to \mathfrak{k}$ is given by $\Psi(|\phi\rangle) = \mu(K.|\phi\rangle) \cap \mathfrak{t}_+$). Nevertheless, the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ has a finite number of points $\{\alpha_i\}$ for which $\Psi^{-1}(\alpha_i)$ contains critical K-orbits. Let C_α denote the set of critical K orbits mapped by Ψ on $\alpha \in \mathfrak{t}_+$ and N_α be the set of all the points that are taken by gradient flow of $-||\mu||^2$ to C_α. The quotient space $N_\alpha /\!\!/ G$, obtained from N_α by dividing N_α by the equivalence relation induced from the gradient flow, and the space C_α/K are isomorphic (see Figure 3). Moreover, they are projective algebraic varieties. It is worth noting that the above discribed construction is analogous to the GIT one. For $\alpha = 0$ we get that $N_0 = \mathbb{P}(\mathcal{H})_{ss}$ and $C_0 = \mu^{-1}(0)$. Using the so defined quivalence relation we can think of a quotient space $\mathbb{P}(\mathcal{H})$ by G (abusing notation: $\mathbb{P}(\mathcal{H})/G$) as of the space consisting of a finite number of projective algebraic varieties:

$$\mathbb{P}(\mathcal{H})/G \cong \bigcup_\alpha N_\alpha /\!\!/ G \cong \bigcup_\alpha C_\alpha/K. \tag{6}$$

The map Ψ has another important property, namely not only $\Psi(\mathbb{P}(\mathcal{H}))$ is a convex polytope but also the image of every G-orbit $\Psi(\overline{G.x})$ has this property [13]. A finite number of varieties C_α/K is the result of the fact that N_α can be equivalently defined as those $x \in \mathbb{P}(\mathcal{H})$ for which polytopes $\Psi(\overline{G.x})$ share the nearest point to the origin. But the momentum map convexity theorem for G-orbits ensures that the number of such polytopes is finite [23], so the number of manifolds C_α is also finite. Summing up we get the following correspondence:

Table 1: A dictionary

G-orbit	SLOCC class of states
the momentum map μ	the map which assigns to a state $[v]$ the collection of its reduced one-particle density matrices
$\|\|\mu\|\|^2([v])$	the total variance of state Var($[v]$), linear entropy
closure equivalence class of orbits	family of asymptotically equivalent SLOCC classes
stable point	SLOCC family consists of exactly one SLOCC class
semistable but not stable point	SLOCC family consists of many SLOCC classes
$\Psi(\overline{G.[v]})$	SLOCC momentum polytope, collection of all possible spectra of reduced one-particle density matrices for $[u] \in \overline{G.[v]}$
strata N_α	group of families of SLOCC classes - all states for which SLOCC momentum polytopes have the same closest point to the origin
C_α	set of critical points of Var($[v]$) with the same spectra of reduced one-particle density matrices

The space $\mathbb{P}(\mathcal{H})$ can be also divided into a finite number of generalized SLOCC classes using polytopes $\Psi(\overline{G.x})$, which in [70] are called entanglement polytopes. This is done by saying that two states are equivalent when their entanglement polytopes are the same.

AD

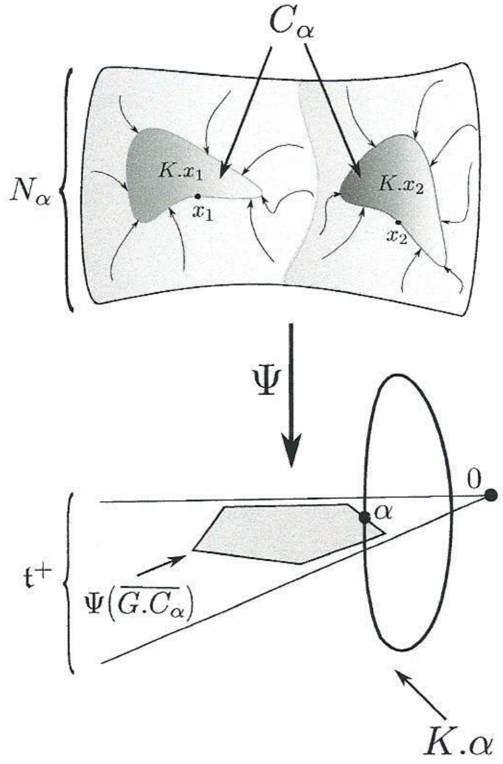

Figure 3: The sets N_α and C_α, with two exemplary critical K-orbits, $K.x_1$ and $K.x_2$. The arrows represent the gradient flow of $-||\mu||^2$.

Decomposition (6) is identical with that division up to the existence of polytopes that have a common closest point to the origin.

The key ingredient needed to obtain decomposition (6) is the knowledge of the critical K-orbits of $||\mu||^2$. In [3] we find them for two distinguishable and indistinguishable particles, three qubits and any number of two-state bosons. For four qubits we show that most classes found in [67] are c-equivalent with the class corresponding to $\mu^{-1}(0)$. Calculations of critical points presented in [3] have been made by a direct application of the definition of a critical point of $||\mu||^2$, i.e. by solving the eigenproblem $\mu([v]).v = \lambda v$. Note that in this equation the matrix $\mu([v])$ depends in a nonlinear way on the vector $[v]$. The above mentioned cases are the only ones for which direct application of the definition allows easy calculation of the critical points.

4.4.1 Critical points of $||\mu||^2$ for many qubits

As we have seen in the previous section, the critical points of $||\mu||^2$, or of the linear entropy, play a key role in understanding the generalised SLOCC classes. Finding the critical states by direct application of the definition is a computationally difficult task, as it requires solving an eigenproblem for a matrix, depending nonlinearly on a vector it acts on. In paper [1] we propose a more tractable method that is based on an interplay between momentum maps for

AJ

abelian and non-abelian Lie groups.

For a compact group K, we denote by T its maximal torus, which is a maximal compact and connected abelian subgroup. For example, when $K = SU(N)$, a maximal torus consists of unitary diagonal matrices with the determinant one. A momentum map $\mu_T : M \to \mathfrak{t}$ for the action of T on M is given by the composition of $\mu : M \to \mathfrak{k}$ with the projection on the Cartan subalgebra $\mathfrak{t} = Lie(T)$. Therefore, we have $\mu([v]) = \mu_T([v]) + \alpha$, where $\alpha \in \mathfrak{t}^{\perp}$. By the convexity theorem, $\mu_T(M)$ is a convex polytope. For abelian groups, the convexity theorem specifies the vertices of the polytope [8]. The vertices are given by the set of weights $\mathbb{A} = \mu_T(M_T)$, where M_T are the fixed points for the action of T on M[3]. The critical points of $||\mu_T||^2$, must satisfy a similar condition as the critial points of $||\mu||^2$, i.e. $\mu_T([v]).v = \lambda v$. Therefore, for $\beta \in \mu_T(M)$ a point $[v]$ is a critical point iff $\beta.v = \lambda v$, i.e. $[v]$ is a fixed point for $T_\beta = \{e^{t\beta} : t \in \mathbb{R}\}$ and $\mu_T([v]) = \beta$. The fixed-point set M_{T_β} for the action of T_β is not a symplectic variety. However, weakening the definition of M_{T_β} by demanding points $[v]$ to satisfy $\langle \mu_T([v]), \beta \rangle = \langle \beta, \beta \rangle$ instead of being mapped to β, we get a symplectic variety Z_β. Points from Z_β are sent by μ_T to the hyperplane that is perpendicular to β and contains β [39]. Clearly, $M_{T_\beta} \subset Z_\beta$. The set Z_β is a T-invariant symplectic variety, hence, by the convexity theorem, we have that $\mu_T(Z_\beta)$ is a convex polytope, which is spanned by a subset of weights from $\mathbb{A}$. The definition of Z_β implies that β is the closest to zero point of this polytope. In other words, $[v] \in M$ is a critical point of $||\mu_T||^2$ iff it is mapped to a minimal convex combination of weights, β, and $[v] \in Z_\beta$ [39].

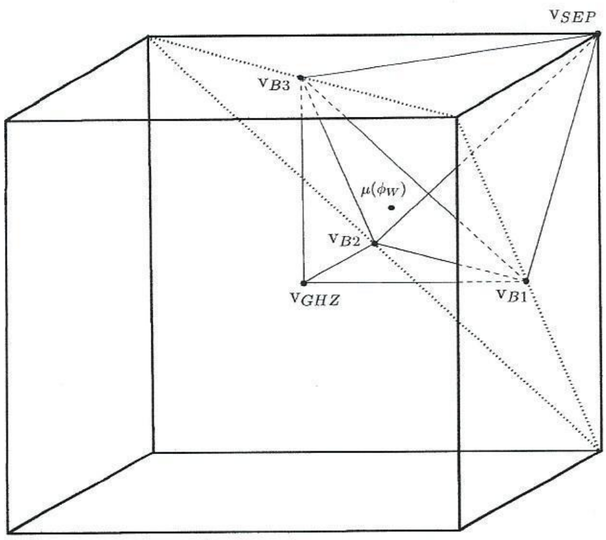

Figure 4: Minimal weight combinations for three qubits. Point v_{GHZ} is the image of state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, pionts v_{Bi} correspond to the biseparable states and $|\phi_W\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle)$.

Function $||\mu||^2$ is K-invariant, therefore we can restrict our consideration to critical points satisfying $\mu([v]) \in \mathfrak{t}_+$. For such states we have $\mu([v]) = \mu_T([v])$ and $[v]$ is a critical point of

[3]A point $x \in M$ is fixed by the action of T iff $\forall t \in T\, t.x = x$

$||\mu||^2$ iff it is a critical point of $||\mu_T||^2$. Let us denote by $\mathcal{B}$ the set of all minimal combinations of weights from $\mathbb{A}$ that belong to $\mathfrak{t}_+$. Then, a state $[v]$ is a critical one if and only if $\mu([v]) \in \mathcal{B}$ and $[v] \in Z_\beta$. Critical sets are therefore of the form $C_\beta = K.(Z_\beta \cap \mu^{-1}(\beta))$, where $\beta \in \mathcal{B}$. In our work [1] we apply the above reasoning in order to compute the critical points of the linear entropy for pure states of L qubits. Set $\mathbb{A}$ is the image under μ of the basis states $B = |i_1, \ldots, i_L\rangle$, where $i_k \in \{0, 1\}$, hence $\#\mathbb{A} = 2^L$. We discuss the algorithm of finding the minimal combinations of weights and list the results up to $L = 5$ (the construction of the set of minimal combinations of weights is shown on figure 4). We also show that for $\beta \in \mathcal{B}$, the set $Z_\beta = \mathbb{P}(\mathcal{S})$, where S is spanned by the basis states whose weights span β. Moreover, we show when sets C_β are nonempty and for each $\beta \in \mathcal{B}$ we describe a construction of a state that is mapped to β. We conclude that the number of critical values of the linear entropy grows super-exponentially with L.

Critical $\alpha \in \Psi(\mathbb{P}(\mathcal{H}))$	State	$\mathrm{E}(\phi)$
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	Sep	0
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	TriSep	$\frac{1}{4}$
$\begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\lvert \mathrm{W}^{(3)}\rangle \otimes \lvert 1\rangle$	$\frac{1}{3}$
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	BiSep	$\frac{3}{8}$
$\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$	W	$\frac{3}{8}$
$\begin{pmatrix} -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}, \begin{pmatrix} -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$	Φ_3	$\frac{9}{20}$
$\begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Φ_2	$\frac{11}{24}$
$\begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$	Φ_1	$\frac{27}{56}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	GHZ	$\frac{1}{2}$

Table 2: One-qubit reduced density matrices for critical states of four qubits. The listed states are: $|\mathrm{TriSep}\rangle = \frac{1}{\sqrt{2}}|11\rangle \otimes (|00\rangle + |11\rangle)$, $|\mathrm{BiSep}\rangle = \frac{1}{\sqrt{2}}|1\rangle \otimes (|000\rangle + |111\rangle)$, $|\mathrm{W}\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$, $|\Phi_3\rangle = \sqrt{\frac{3}{10}}(|1101\rangle + |1110\rangle) + \sqrt{\frac{2}{5}}|0011\rangle$, $|\Phi_2\rangle = \frac{1}{2\sqrt{3}}(|1011\rangle + |1110\rangle) - \frac{1}{2}(|0101\rangle + |0011\rangle) + \frac{1}{\sqrt{3}}|0110\rangle$, $|\Phi_1\rangle = \sqrt{\frac{3}{14}}(|0011\rangle + |0101\rangle + |1001\rangle) + \sqrt{\frac{5}{14}}|1110\rangle$, $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$.

4.5 Geometric and topological characterization of CQ and CC sates

In previous chapters we discussed applications of the momentum map in two significant problems of the theory of quantum corellations: K and $K^{\mathbb{C}}$ equivalence of pure states. Mixed states are subject of publication [2], where we discuss geometric and topological aspects of quantum correlations that exist for separable (non-entangled) states. The existence of quantum correlations for multipartite separable mixed states can be regarded as one of the most interesting quantum information discoveries of the last decade. In 2001 Ollivier and Żurek [56] and independently Henderson and Vedral [30] introduced the notion of quantum discord as a measure of the quantumness of correlations. Quantum discord is always non-negative [19]. The states with vanishing quantum discord are called *pointer states.* They form the boundary between classical and quantum correlations [19]. Bipartite pointer states can be identified with the so-called classical-quantum, CQ states [19]. An important subclass of CQ states are classical-classical, CC states [43].

For $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathbb{C}^{N_1}$ and $\mathcal{H}_B = \mathbb{C}^{N_2}$ a state is CC if it can be written as $\rho = \sum_{i,j} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$, where $\{|i\rangle\}_{i=1}^{N_1}$ is an orthonormal basis in $\mathcal{H}_A$ and $\{|j\rangle\}_{j=1}^{N_2}$ in $\mathcal{H}_B$. A state ρ is a CQ state if it can be written as $\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$, where $\{\rho_i\}_{i=1}^{N_2}$ are the density matrices on $\mathcal{H}_B$. Both classes are of measure zero in $\mathcal{H}$ [21]. For pure states, separable states are exactly zero-discord states. It was shown in [63] that pure saprable states are geometrically distinguished in the state space and belong to the unique symplectic K-orbit in $\mathbb{P}(\mathcal{H})$. For mixed states, already for two particles it is easy to see that there are infinitely many symplectic K-orbits and there are separable states through which K-orbits are not symplectic. Thus a simple extension of the results of [63], even for two-particle mixed states is not possible. In [2] we show four facts that are geometric and topological characterisations of CC and CQ states that extend results of [63] to mixed states: (1) the set of CQ states is the closure of all symplectic orbits of $K = SU(N_1) \times I_{N_2}$ action, (2) the set of CC states is the closure of all symplectic orbits of $K = SU(N_1) \times SU(N_2)$ action, (3) the set of CQ states is exactly the set of $K = SU(N_1) \times I_{N_2}$ orbits whose Euler-Poincaré characteristics χ do not vanish, (4) the set of CC states is exactly the set of $K = SU(N_1) \times SU(N_2)$ orbits whose Euler-Poincaré characteristics χ do not vanish.

The space of all density matrices is not a symplectic space (the symplectic form is degenerate). Nevertheless, the set of density matrices with the fixed spectrum $\mathcal{O}_\rho$, which is the adjoint orbit of $SU(\mathcal{H})$ through ρ is symplectic. Therefore, the action of the above given groups K on $\mathcal{O}_\rho$ leads to existence of the momentum map $\mu : \mathcal{O}_\rho \to \mathfrak{k}$ [24]. In order to check if a given orbit $K.\sigma$ (the action of K on $\sigma \in \mathcal{O}_\rho$ is the adjoint action) is or is not symplectic it is enough to consider the restriction of the momentum map μ to $K.\sigma$. Then $K.\sigma$ is symplectic if this restriction is bijective. The computational conditions for μ to be bijective are given in the Kostant-Sternberg theorem [44] which we used in publication [2]. Let us note that since $K.\rho$ is mapped by μ onto adjoint orbit in $\mathfrak{k}$, non-symplecticity of $K.\rho$ (the degeneration of symplectic form on $K.\rho$) can be measured as $D(K.\rho) = \dim K.\rho - \dim Ad_K \mu(\rho)$. For two qubits the CC states, in a fixed basis, form a 3-dimensional simplex and therefore it is possible to see how the closure of the symplectic $K = SU(N_1) \times SUN_2$ orbits forms the set of CC states (figure. 5, 6 and 7). In [2] we also discuss existence of Kähler structure and show that it is present on all considered symplectic K-orbits.

For finding Euler-Poincaré characteristics χ we use the Hopf-Samelson theorem [32]. This theorem says that for action of a compact group K on a manifold M the Euler-Poincaré characteristics χ of the orbit K/K_x passing through $x \in M$ is given by:

1. If the maximal torus T of K is contained in K_x then $\chi(K/K_x) = \frac{|W_K|}{|W_{K_x}|}$, where W_K and

AD

W_{K_x} are Weyl groups of K and K_x respectively.

2. Otherwise, $\chi(K/K_x) = 0$.

In publication [2] we show that orbits of the discussed groups through CC and CQ states are the only orbits with stabiliser subgroups containing maximal torus. We also calculate ranks of the Weyl groups and obtain formula for χ.

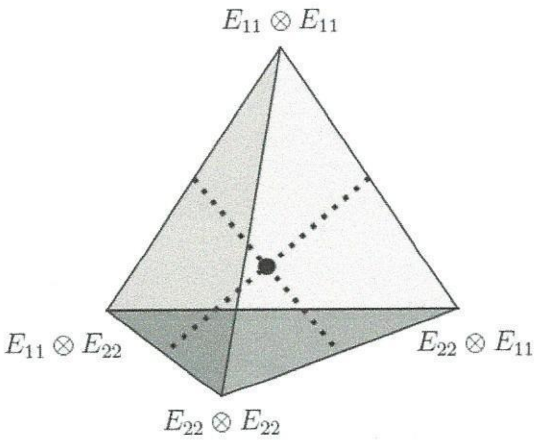

Figure 5: Dimensions of orbits through CC states of two qbits. The large dot: dim $K.\rho = 0$, the dotted lines: dim $K.\rho = 2$, elsewhere: dim $K.\rho = 4$.

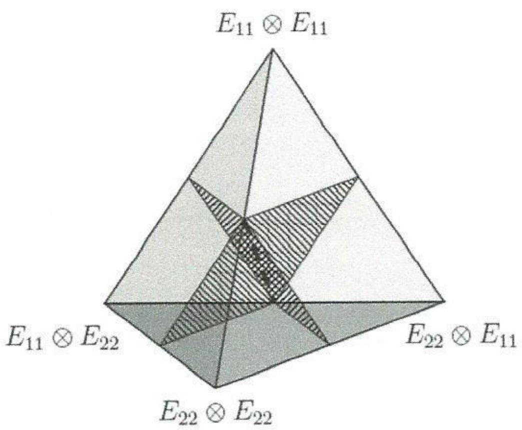

Figure 6: Ranks of $\omega|_{K.\rho}$ for orbits through CC states of two qbits. The thick dashed line: rk $\omega|_{K.\rho} = 0$, the lined surfaces: rk $\omega|_{K.\rho} = 2$, elsewhere: rk $\omega|_{K.\rho} = 4$.

AJ

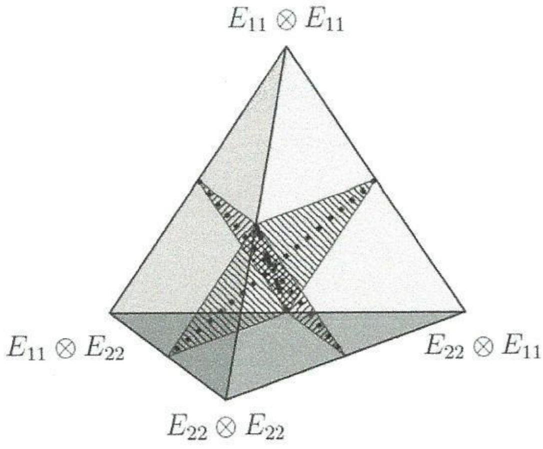

Figure 7: Degrees of degeneracy of $\omega|_{K.\rho}$ for orbits through CC states of two qbits. The thick dashed line: $\mathrm{D}(K.\rho) = 4$, the lined surfaces: $\mathrm{D}(K.\rho) = 2$, the dotted lines and elsewhere: $\mathrm{D}(K.\rho) = 0$.

5 Discussion of other scientific accomplishments

My other scientific accomplishments concern: (1) scattering from isospectral graphs, (2) classification of abelian quantum statistics on quantum graphs, (3) (non)integrability of Hamiltonian systems on topologically nontrivial phase spaces, (4) universality in linear quantum optics, (5) geometric measures of entanglement. In the following I briefly describe this subjects.

Scattering from isospectral graphs

In 1966 Mark Kac asked his famous question 'Can one hear the shape of a drum?' [35]. This question can be reformulated as: Does the Laplacian defined on a planar region in $\mathbb{R}^2$ with Dirichlet boundary conditions have a unique spectrum?. The answer to this question was found only in 1992, when authors of [22] constructed a pair of isospectral domains in $\mathbb{R}^2$. A quantum graph consist of egdes called bonds and vertices in which bonds connect. By fixing boundary conditions at the vertices we obtain a selfadjoint Laplace operator on a graph, and therefore we can ask the Kac's question. The authors of [26] proved that one can hear the shape of a quantum graph provided the bonds lengths are incommensurate. Next in 2008 the method to construct isospectral graphs, based on the representation theory of finite groups was given [9]. In 2009 during my intership at the Weizmann Institute in Israel I showed that scattering matrices of isospectral graphs have identical spectra and distributions of poles. These kind of graphs are called isoscattering. I also gave the first and only known up to now method for construction of isoscattering graphs. The results were published together with my collaborators [10]. Experimental verification of this theoretic results was carried in the group of Prof. L. Sirko with whom I closely collaborated in this respect [47, 48].

Classification of abelian quantum statistics on quantum graphs

In 1977 Leinaas and Myrheim showed that Abelian representations of the fundamental group of the classical configuration space determine the possible realisations of quantum statistics. If the fundamental group is the group of permutations then the topological approach to quantum statistics coincides with the standard one, namely we have two exchange statistics: Bose and Fermi, respectively. When the space dimension is equal to two (for the particles on $\mathbb{R}^2$), the fundamental group is not the permutation group but a braid group which has infinitely many elements. The abelian version of the braid group is isomorphic with the group of integers $\mathbb{Z}$, which means that exchanging particles can lead to any phase. For quantum graphs that are locally one-dimensional the problem of quantum statistics turns out to be very interesting and also far more difficult. During my stay at the University of Bristol in the period 10/2010-10/2013 I managed to (1) provide the full classification of all possible Abelian quantum statistics on any connected simple graph (publication [28]), (2) to propose a construction method of discrete Morse functions for two particle configuration spaces [62].

As I showed, the number of anyon phases is determined by the connectivity of the considered graph. For 3-connected nonplanar graphs only possible statistics are bosons and fermions. For 3-connected planar graphs there is exactly one anyone phase. Thus, we can say that from the point of view of topology, up to the first homology group, 3-connected graphs behave like $\mathbb{R}^2$ when planar and $\mathbb{R}^3$ otherwise. Moreover, the number of anyon phases does not depend on the number of particles for graphs that are at least 2-connected. Interestingly, these graphs may have more than one anyon phase, and their number is determined by the number of 3-connected components in a decomposition of the 2-connected graph. Although this decomposition is not unique, the number of components obtained is always the same. For 1-connected graphs quantum statistics depends on the number of particles in the system. All of these results were obtained by developing a new set of methods for calculating the homology groups, which bring together some known facts from graph theory, discrete Morse theory and simple calculations for certain small graphs. The results of [28] and [62], published in Communications in Mathematical Physics and Journal of Physics A, were the basis of my doctoral dissertation in mathematics defended at the University of Bristol in 2014.

(Non)integrability of Hamiltonian systems on topologically nontrivial phase spaces

One of the problems in quantum chaos theory is showing that the classical limit of quantum chaotic system is classically chaotic as well. The more modest goal would be to prove that the limit is not integrable. In publication [60], which was the basis of my master thesis defended in February 2010 at the University of Warsaw, I presented the first analytical proof of nonitegrability of a Hamiltonian system with a symmetry group $SU(3)$, which is a classical limit of quantum chaotic system. The considered system was defined on the dual space to the Lie algebra $\mathfrak{su}(3)$, which is naturally a Poisson manifold. To prove nonintegrability I used a recently developed theory of Morales and Ramis [51] that is based on differential Galois theory. The Hamilton equations are partial differential equations. Therefore to apply differential Galois theory one has to find some ordinary differential equation related to Hamilton equations. In short words, the problem boils down to finding a special solution of Hamilton equations and then linearising them around this solution. Morales-Ramis theory ensures that if the linearised system of ordinary differential equations has non-abelian differential Galois group the corresponding Hamiltonian system is not integrable. In paper [60] I showed that for the considered linearization the differential Galois group is not solvable and hence it is not abelian.

Universality in linear quantum optics

One of the fundamental problems in quantum linear optics is construction of n-mode gates. About 20 years ago the authors of [59] showed that having at the disposal all possible 2-mode gates and phase shifters, one can construct any n-mode gate. This result, however, is of limited practical significance because typically we have only access to a few types of optical gates and from them we want to build another ones. Mathematically, the considered problem reduces to determination of the set generated by some elements of a Lie group. If this set is a dense subset of the group we call it universal. The universal set of gate types is a set that allows the construction of any optical gate with an arbitrary precision. In publication [65] I am dealing with the problem of generating $SO(N)$ groups having at disposal only one optical gate that operates on two or three modes, thus belonging to the group $SO(2)$ or $SO(3)$. Using tools of control theory I show that a nontrivial 2-mode gate is always universal. The same is true for almost all 3-mode gates.

Geometric measures of entanglement

Separability of a multipartite quantum state is invariant under the action of the certain transformations allowed by quantum mechanics. From a mathematical point of view, this situation can be described by the action of a compact Lie group K on a manifold M. The considered manifold of course depends on the particular physical situation. For example, for pure states it is a complex projective space $M = \mathbb{P}(\mathcal{H})$. Interestingly, M is equipped with a symplectic structure induced by the natural symplectic structure existing on each complex Hilbert space. Orbits of K-action on M being submanifolds of M may also, under certain conditions, inherit symplectic structure, and even the complex structure of the Hilbert space $\mathcal{H}$. In the paper published in Communications in Mathematical Physis [63], which was the basis of my doctoral dissertation defended in 2011 at the University of Warsaw, I showed that entanglement between various subsystems can be quantitatively characterised by the dimension of the degeneration of the canonical symplectic form restricted to K orbits. The paper [61] is a continuation of [63] and gives a geometric characterization of low-dimensional local unitary orbits by showing that for the system of two identical but distinguishable particles the generic K-orbit is coisotropic.

References

[1] T. Maciążek, A.Sawicki, Critical points of the linear entropy for pure L-qubit states J. Phys. A: Math. Theor. 48 045305, 2015

[2] M. Oszmaniec, P. Suwara, A. Sawicki, Geometry and topology of CC and CQ states, J. Math. Phys. 55, 06220, 2014

[3] A. Sawicki, M. Oszmaniec, M. Kuś, Convexity of momentum map, Morse index, and quantum entanglement Reviews in Mathematical Physics, 26, 1450004, 2014

[4] T. Maciążek, M. Oszmaniec, A. Sawicki, 2013, How many invariant polynomials are needed to decide local unitary equivalence of qubit states?, J. Math. Phys. 54, 092201, 2014

AJ

[5] A. Sawicki A., V. V. Tsanov, A link between quantum entanglement, secant varieties and sphericity, J. Phys. A: Math. Theor. 46 265301, 2013

[6] A. Huckleberry, M. Kuś, A. Sawicki, Bipartite entanglement, spherical actions and geometry of local unitary orbits, J. Math. Phys. 54, 022202, 2013

[7] D. Akhiezer and E. B. Vinberg, Weakly symmetric spaces and spherical varieties, Trans. Groups 4, 3–24, 1999

[8] M. F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14, 1–15, 1982

[9] R. Band, O. Parzanchevski, and G. Ben-Shach, The isospectral fruits of representation theory: Quantum graphs and drums, J. Phys. A: Math. Theor., vol. 42, p. 175202, 2009.

[10] R Band, A. Sawicki A., U. Smilansky, Scattering from isospectral quantum graphs. J. Phys. A: Math. Theor. 43 415201, 2010

[11] A. Barut and B. Rączka, Theory of group representations and applications, PWN, Warszawa, 1980.

[12] Bredon, G. E., Introduction to compact transformation groups, Pure and Applied Math. 46, Academic Press, 1972

[13] M. Brion, Sur l'image de l'application moment, Lect. Notes Math. 1296, 177–192, 1987

[14] J. Buczyński and J. M. Landsberg, Rank of tensors and a generalization of secant varieties, Linear Algebra Appl. 438, no. 2, 668–689, 2013

[15] E. Carlini, M. V. Catalisano, A. V. Geramita, The solution of the Waring problem for monomials and the sum of coprime monomials, J. Algebra 370, 5–14, 2012

[16] L. Chen, D. Z Djokovic, M. Grassl, B. Zeng, Canonical form of three-fermion pure-states with six single particle states, J. Math. Phys. 55, 082203, 2014

[17] D. Chruściński, Symplectic orbits in quantum state space, J. Math. Phys. 31 1587, 1990

[18] G. Comas, M. Seiguer, On the rank of a binary form, Found. Comput. Math. 11 , p. 65–78, 2011

[19] A. Datta, Condition for the Nullity of Quantum Discord, arXiv:1003.5256, 2011

[20] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A 62, 062314, 2000

[21] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. AcĂn ,Almost all quantum states have nonclassical correlations, Phys. Rev. A 81, 052318, 2010

[22] C. Gordon, D. Webb, and S. Wolpert, One cannot hear the shape of a drum, Bull. Am. Math. Soc., vol. 27, pp. 134-38, 1992

[23] V. Guillemin and R. Sjamaar, Convexity theorems for varieties invariant under a Borel subgroup, Pure Appl. Math. Q. 2, no. 3, 637–653, 2006

[24] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge ,1984

[25] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67, 491513, 1982

[26] B. Gutkin and U. Smilansky, Can one hear the shape of a graph?, J. Phys. A, vol. 34, p. 6061, 2001

[27] B. C. Hall, Lie groups, Lie algebras, and representations, an elementary introduction, Springer, New York, 2003.

[28] J. Harrison, J. Keating, J. Robbins, A. Sawicki, n-Particle Quantum Statistics on Graphs, Commun. Math. Phys. 330, 1293-1326, 2014

[29] P. Heinzner, A. Huckleberry, Kählerian potentials and convexity properties of the moment map, Invent. Math. 126, 6584, 1996

[30] L. Henderson and V. Vedral, Classical, quantum and total correlations, Journal of Physics A 34, 6899, 2001

[31] A. Higuchi, A. Sudbery, J. Szulc, One-qubit reduced states of a pure many-qubit state: polygon inequalities, Phys. Rev. Lett. 90, 107902, 2003

[32] H. Hopf, H. Samelson, Ein Satzüber die Wirkungsräume geschlossener Liescher Gruppen, Comment. Math. Helv. 13, 240, 251 (German), MR 0006546, 1941

[33] R. Horodecki et. al., Quantum entanglement, Rev. Mod. Phys. 81, 865942 (2009)

[34] A. T. Huckleberry and T. Wurzbacher, Multiplicity-free complex manifolds, Mathematische Annalen 286, 261–280, 1990

[35] M. Kac, Can one hear the shape of a drum?, Am. Math. Mon., 73,1-23, 1966

[36] V. Kac, Some remarks on nilpotent obits, J. of Algebra 64, 190–213, 1980

[37] G. Kempf, L. Ness, The lengths of vectors in representation spaces, Lectures Notes in Mathematics, 732, 233–243 1982

[38] F. C. Kirwan, Convexity properties of the moment mapping, Invent. Math. 77, 547552, 1984

[39] F. C. Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes 31, Princeton, NJ: Princeton University Press, 1982

AJ

[40] A. Klyachko, Quantum marginal problem and representation of the symmetry group, arXiv:quant-ph/0409113

[41] A. Klyachko, Dynamic symmetry approach to entanglement, Proceedings of the NATO Advanced Study Institute on Physics and Theoretical Computer Science, IOS Press, Amsterdam, 2007

[42] F. Knop, Some remarks on multiplicity free spaces, Representation theories and algebraic geometry (Montreal, PQ, 1997), 301317, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 514, Kluwer Acad. Publ., Dordrecht, 1998

[43] J. K. Korbicz, P. Horodecki, R. Horodecki, Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains, Phys. Rev. A 86, 042319, 2012

[44] B. Kostant and S. Sternberg, Symplectic projective orbits. New directions in applied mathematics, papers presented April 25/26, 1980, on the occasion of the Case Centennial Celebration (Hilton, P. J. and Young, G. S., eds.) New York: Springer, 81-84, 1982

[45] B. Kraus, Local Unitary Equivalence of Multipartite Pure States, Phys. Rev. Lett. 104, 020504, 2010

[46] J. M. Landsberg,Tensors: Geometry and Applications, AMS, Providence, RI, 2012

[47] M. Ławniczak, A. Sawicki. S. Bauch, M. Kuś, L. Sirko, Resonances and poles in isoscattering microwave networks and graphs, Phys. Rev. E 89, 032911, 2014

[48] O. Hul, M. Ławniczak, S. Bauch, A. Sawicki, M. Kuś, L. Sirko, Are scattering properties of graphs uniquely connected to their shapes?, Phys. Rev. Lett. 109, 040402, 2012

[49] E. Meinrenken, C. Woodward, Moduli spaces of flat connections on 2-manifolds, cobordism, and Witten's volume formulas, Advances in geometry, Progr. Math. 172, 271-295, 1999

[50] J. S. Milne, Algebraic geometry, Taiaroa Publishing Erehwon, Version 5.00, 2005

[51] J. J. Morales-Ruiz, Differential Galois Theory and Non-Integrability of Hamiltonian Systems, Birkhauser, Basel, 1999

[52] S. Mukai, An Introduction to Invariants and Moduli, Cambridge Studies in Advanced Mathematics, No. 81, 2003

[53] D. Mumford, Stability of projective varieties, L'Enseignement Mathématique, 1977

[54] L. Ness, A stratification of the null cone via the moment map [with an appendix by D. Mumford], Amer. J. Math. 106(6), 1281–1329, 1984

[55] M.A Nielsen and I.L Chuang, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge, 2000

[56] H. Ollivier and W. H. Żurek, Quantum Discord: A Measure of the Quantumness of Correlations, Physics Review Letters vol. 88, 01790, 2001

[57] M. Oszmaniec and M. Kuś, On detection of quasiclassical states, J. Phys. A 45, 244034, 2012

[58] A. M. Perelomov, Generalized Coherent States and Their Applications, Springer, Berlin, 1986

[59] M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani, Experimental realization of any discrete unitary operator, Physical Review Letters, 73(1):58–61, 1994.

[60] A. Sawicki, M. Kuś, Classical nonintegrability of a quantum chaotic SU(3) Hamiltonian system, Physica D: Nonlinear Phenomena, 239 719-726, 2010

[61] A. Sawicki, M. Kuś, Geometry of the local equivalence of states, J. Phys. A: Math. Theor. 44 49530, 2011

[62] A. Sawicki, Discrete Morse functions for graph configuration spaces, J. Phys. A: Math. Theor. 45 505202, 2012

[63] A. Sawicki, A. Huckleberry, and M. Kuś, Symplectic geometry of entanglement. Comm. Math. Phys., 305(2):441–468, 2011

[64] A. Sawicki, M. Walter, M. Kuś, When is a pure state of three qubits determined by its single-partite reduced density matrices?, J. Phys. A: Math. Theor. **46**, 055304, 2013

[65] A. Sawicki, Universality of beamsplitters, Quantum Information and Computation 16, 3 and 4, 0291-0312, 2016

[66] A. Sudbery, On local invariants of pure three-qubit states, J. Phys. A , 34, 643, 2001

[67] F. Verstraete, J. Dehaene, and B. De Moor, Normal forms and entanglement measures for multipartite quantum states, Phys. Rev. A 68, 012103, 2003

[68] G. Vidal, Entanglement monotones, Journal of Modern Optics 47, 1999

[69] P. Vrana, Local unitary invariants for multipartite quantum systems, J. Phys. A 44, 115302, 2011

[70] M. Walter, B. Doran, D. Gross, and M. Christandl, Entanglement Polytopes, Science 340, 1205, 2013

Adam Sawicki 23.03.2016