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Autopresentation

1 Name and last name

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2 Certificates and scientific degrees

— M. Sci study: 1988 – 1993, theoretical physics, Nicolaus Copernicus University,

— M. Sci degree: July 1993, physical science, Nicolaus Copernicus University, M. Sci thesis: *Statistical properties of the Heisenberg Hamiltonian spectrum* (supervisor: Prof. Dr J. Karwowski)

— PhD: November 2000, physical science, Nicolaus Copernicus University, PhD thesis: *Contact transformations and their applications in thermodynamics of quasi-equilibrium processes* (supervisor: Dr R. Mrugała)

3 Academic appointments

— 1.09.1993 – 2001: Research Assistant at Institute of Physics N. Copernicus University,

— 2001–01.2012: Assistant Professor (Adiunkt) in Department of Physics, Astronomy and Informatics, Nicolaus Coopernicus University,

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4 Awards and prizes

(1) Team Award of second degree of the Rector of Nicolaus Copernicus University for the preparation and organization of *38 Symposium on Mathematical Physics: Quantum Entanglement and Geometry*

(2) Individual Award of fourth degree of Rector of Nicolaus Copernicus University for achievements obtained in teaching organization process in 2007, granted at 17.11.2008

(3) Team Award of third degree of Rector of NCU for scientific achievements in 2008, awarded at 17.11.2009

(4) Team Award of second degree of Rector of NCU for scientific achievements in 2010.

5 Achievement indicated in accordance with the Art. 16 Item 2 of the Act of 16 March 2003 on Academic Degrees and Title and Degrees and Title in Art (Journal of Laws No. 65, item. 595 as amended).

Monograph (in polish) under the title

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My scientific research and interest after obtaining a PhD focused on three topics:

(1) quantum correlations, in particular, the role of quantum entanglement and discord in their quantifying,

(2) using Lie-algebraic methods in solving ordinary differential equations,

(3) quantisation of dissipative systems.

As a summary of my scientific activity in the subject of quantum correlations, I presented the monograph *Nonclassical correlations. Quantum entanglement and discord* [**?**]. The monograph consists of four chapters, which include the most important results of my research presented in the wider context of contemporary problems of classification, qualitative and quantitative evaluation of the classical and quantum correlations and their role in quantum computing.

The well-known manifestation of quantum correlations in composite systems is *entanglement* [1]. For a pure state of two parties $|\Psi\rangle$ entanglement means the lack of factorization of $|\Psi\rangle$ into a product of vectors of subsystems. Hence, such a state is characterised by a linear combination of product states, i.e.,

$$|\Psi\rangle = \sum_{jk} c_{jk} |\phi_j\rangle \otimes |\psi_k\rangle$$

where the coefficients c_{jk} do not factorize. For a mixed state ρ_{AB} the situation becomes more subtle and whether it is entangled or not (no entanglement = separability) is determined to a large extent by the way it is produced. To be precise, ρ_{AB} is *separable* if there exist in distant laboratories A and B two families of states $\{\rho_k^A\}$, $\{\rho_k^B\}$, respectively, prepared by local operations and classical communication (e.g., transmission of probability distribution $\{p_k\}$) between A and B *only* and such that

$$\rho_{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B \tag{1}$$

remains untouched by the interaction with environment.

In [2, Sect. 3] I formulated the following crucial problems of entanglement analysis:

(1) How to detect entanglement, i.e., how to distinguish the entangled states of a composite system from separable ones? The answer to this question is known for pure states but for mixed states the simple solution is possible only for low-dimensional systems described in Hilbert spaces $\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2$ (for example, two spin-1/2 particles) or $\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^3$. For particles with spins higher than 1/2, we must rely on numerical algorithms. From the other hand, it should be developed methods that can detect entangled states experimentally using available measurement techniques.

(2) How to measure entanglement? Again, methods using entropy-like quantities characterising a given state provide simple answer only for pure states. Generalizations of such entanglement measures for mixed states are in practice difficult to handle. Therefore, some methods for estimating the entanglement measures should be developed.

(3) How to protect entanglement? It turned out that quantum correlations are very gentle, when the state undergoes even a weak interaction with environment. While coherences usually delay exponentially in time, entanglement can completely disappear in finite time. Hence, it is important to work out some methods of entanglement activation and distillation.

Note however, that entanglement is just one of many possible manifestations of non-classical correlations between states or, more generally, systems. These non-classical correlations can account for unique and special resource which can be used in a number of information-theoretic applications (such as the kinetic energy of the system can be used to perform useful work) such as teleportation, encoding, cryptography, and other quantum protocols.

The variety of situations in which correlations in mixed quantum states are revealed, is not surprising, but until recently it was thought that separable states (1) do not contain any useful quantum correlations. But the states $\{\rho_k^A\}$ and/or $\{\rho_k^B\}$ do not have to commute and, therefore, they can contribute correlations into ρ_{AB} arising from their noncommutativity. In this context, only the states

$$\rho_{cc} = \sum_{i,k} p_{ik} |e_i\rangle\langle e_i| \otimes |f_k\rangle\langle f_k| \,, \tag{2}$$

where $\{|e_i\rangle\}$ and $|f_k\rangle$ are orthonormal bases in subsystems A and B, respectively, actually do not contain non-classical correlations. Such states are called *classical* and they are completely characterized by the joint probability distribution $\{p_{ik}\}$. In opposition to classical states, quantum-classical and classical-quantum states of the form

$$\rho_{qc} = \sum_k p_k \rho_k^A \otimes |f_k\rangle\langle f_k| \,, \qquad \rho_{cq} = \sum_i q_i |e_i\rangle\langle e_i| \otimes \rho_i^B \,. \tag{3}$$

can be non-classically correlated if the families of matrices $\{\rho_k^A\}$ or $\{\rho_k^B\}$ do not commute.

All the above mentioned families of states are separable, hence no entanglement measure can classify their quantum correlations. This confirms the previous observation that entanglement reflects only some features of non-classical correlations of quantum states and to distinguish between them one considers various sorts of correlation measures, in particular various kinds of *quantum discord* and its geometric counterpart. More comprehensive discussion of these quantities can be found in [3] and [2, Sect. 2.5 and Sect. 4]. Apart from the classifying role they have also physical meaning in terms of thermodynamics of microscopic systems, quantum information deficit, irreversibility of entanglement, quantum protocols and of completely positive dynamics of open systems (see review article [4] for details).

The information-theoretic point of view resulted in two independent definitions of quantum discord raising from the following observations:

(1) two classical entropic quantities defined as

$$I_{AB} = S(A) + S(B) - S(A,B)\,, \qquad J_{AB} = S(A) - S(A|B)\,, \tag{4}$$

where $S(X)$ is the Shannon entropy of the subsystem X and $S(A|B)$ is the conditional entropy, are equal. But they are not when considering quantum theory with the Shannon entropy S replaced by von Neumann ones H and interpreting the conditional entropy $S(A|B)$ as the Holevo quantity corresponding to the optimal quantum measurement[1]

$$J_{AB} \;\longrightarrow\; C_{AB} = \sup_{\Pi^B} \Big(H(A) - H(A|\{\Pi^B\}) \Big)\,.$$

Taking into account that I_{AB} is interpreted as a measure of total correlations in the state (classical and quantum), non-zero difference $I_{AB} - C_{AB}$ indicates the presence of quantum correlations.

(2) it will be of great importance to have a unique method of distinguishing between classical and quantum correlations. In this context, the quantity

$$D_{AB} := I_{AB} - C_{AB}\,, \tag{5}$$

called *a quantum discord*, is a good candidate for a measure of quantum correlations.

Entanglement, quantum discord and other quantities related to non-classical correlations are intensively studied theoretically as well as experimentally. My studies in this subject correspond to the following problems:

(1) Detecting entanglement in a family of circulant states: [2, Sect. 3.3.4], as well as [7].

(2) Determining local numerical ranges for circulant operators: [2, Sect. 1.6] and [8, 9].

(3) Analysis of SPPT states: [2, Sect. 3.3.3], as well as [5, 6].

(4) Estimating concurrence using entanglement witnesses: [2, Sect. 3.5] and [11, 12].

(5) Study of quantum q-discord based on Tsallis entropy function: [2, Sect. 4.4] and [13, 14, 15].

[1] Suppose that correlations between subsystems A and B are determined by a measurement process on the subsystem B described by a one-parameter family of orthonormal projectors $\{\Pi^B_k\}$. As a result one obtains an ensemble $\{p_k, \rho^A_k\}$, where

$$\rho^A_k = \frac{1}{p_k} \mathrm{tr}_B[(\mathbb{1} \otimes \Pi^B_k)\rho_{AB}(\mathbb{1} \otimes \Pi^B_k)^\dagger]\,,$$

and $p_k = \mathrm{tr}[(\mathbb{1} \otimes \Pi^B_k)\rho_{AB}]$ is a probability of states ρ^A_k to be measured. One can use the *Holevo quantity*

$$\chi(\{p_k, \rho^A_k\}) := \sum_k p_k H(\rho^A_k)$$

corresponding to the ensemble $\{p_k, \rho^A_k\}$ to provide the quantum counterpart of conditional entropy $H(A|\{\Pi^B\})$, this time *conditioned by the measurement*. After optimisation over measurement techniques classical correlations obtained this way read

$$C_{AB} = \sup_{\Pi^B} C^{\Pi^B}_{AB} = \sup_{\Pi^B} \Big(H(A) - H(A|\{\Pi^B\}) \Big) = H(A) - \sup_{\Pi^B} \sum_k p_k H(\rho^A_k)\,.$$

The quantity C_{AB} should replace J_{AB} in (4).

These issues will be discussed in more details in what follows. In addition to the subjects indicated above my monograph provides an overview of

— in Chapter 1 — some ingredients of a mathematical formalism used in the monograph with particular emphasis on the role of entropy and positive mappings,

— in Chapter 2 — classification of non-classical correlations and their measurement,

— in Chapter 3 — quantum entanglement of states, methods of its detection and quantitative evaluation, with particular emphasis on measures and their estimates,

— in Chapter 4 — quantum discord and methods of its measurement.

To my best knowledge there is no similar study in polish literature that would combine different aspects of non-classical correlations and characterize them in a uniform framework.

5.1 Detecting entanglement in a family of circulant states

Circulant states correspond to the following representation of a Hilbert space of a composite system

$$\mathbb{C}^d \otimes \mathbb{C}^d = \bigoplus_{k=1}^{d} \Sigma_k \tag{6}$$

in terms of a direct sum of subspaces Σ_k, where $\Sigma_k = (\mathbb{1} \otimes S^k)\Sigma_1$, $S|e_i\rangle = |e_{i+1}\rangle \bmod d$, $\{|e_1\rangle, \ldots, |e_d\rangle\}$ is a given basis in $\mathbb{C}^d$ and $\Sigma_1 = \mathrm{span}\{|e_1\rangle \otimes |e_1\rangle, |e_2\rangle \otimes |e_2\rangle, |e_3\rangle \otimes |e_3\rangle\}$ (cf. [2, p. 118]).

In my investigations I considered the $3 \otimes 3$-circulant states corresponding to a Hilbert space decomposition (6) given by

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3 \,,$$

where

$$\begin{aligned}
\Sigma_1 &= \mathrm{span}\{|e_1\rangle \otimes |e_1\rangle, |e_2\rangle \otimes |e_2\rangle, |e_3\rangle \otimes |e_3\rangle\} \,, \\
\Sigma_2 &= \mathrm{span}\{|e_1\rangle \otimes |e_2\rangle, |e_2\rangle \otimes |e_3\rangle, |e_3\rangle \otimes |e_1\rangle\} = (\mathbb{1} \otimes S)\Sigma_1 \,, \\
\Sigma_3 &= \mathrm{span}\{|e_1\rangle \otimes |e_3\rangle, |e_2\rangle \otimes |e_1\rangle, |e_3\rangle \otimes |e_2\rangle\} = (\mathbb{1} \otimes S^2)\Sigma_1
\end{aligned}$$

are three orthogonal subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^3$. $3 \otimes 3$-*circulant state* is a sum of three positive semidefinite operators

$$\varrho_k = \sum_{i,j=1}^{3} a_{ij}^{(k)} |i\rangle\langle k| \otimes |i+k\rangle\langle j+k| \,, \quad k = 1, 2, 3 \,,$$

defined on subspaces $\Sigma_1, \Sigma_2, \Sigma_3$, respectively, with positive semidefinite matrices $A^{(k)} = [a_{ij}^{(k)}]$, $k = 1, 2, 3$. Hence the circulant state

$$\varrho = \frac{1}{N}(\varrho_1 + \varrho_2 + \varrho_3) \,,$$

where $N = \mathrm{tr}(A^{(1)} + A^{(2)} + A^{(3)})$ is the normalisation constant, is completely characterised by matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$.

The problem of characterisation of the separability conditions for circulant states is, in general, not solved yet, but some of their ingredients are discussed in [7], as well as in [2,

Sect. 3.3.4]. In these articles I investigated a subfamily of $3 \otimes 3$-circulant corresponding to the following choice of $A^{(k)}$:

$$A^{(1)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} d_{12} & \cdot & \cdot \\ \cdot & d_{23} & \cdot \\ \cdot & \cdot & d_{31} \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} d_{13} & \cdot & \cdot \\ \cdot & d_{21} & \cdot \\ \cdot & \cdot & d_{32} \end{bmatrix}.$$

Consider the family of states $\varrho(A, \varepsilon)$, with $A \equiv A^{(1)} \geqslant 0$, $0 < \varepsilon \neq 1$ and

$$d_{i,i+1} = \varepsilon |a_{i,i+2}|\,, \quad d_{i,i+2} = \frac{1}{\varepsilon} |a_{i,i+2}|\,, \quad i = 1, 2, 3\,.$$

My numerical studies based on generalised realignment criterion CCNR (cf. [2, p. 82])[2], indicate some connection between the notion of a diagonal domination of a matrix A and its separability.[3] Unfortunately, there is no proof (but also no counterexample) of the hypothesis:

HYPOTHESIS 1:

(1) If the matrix A has a strictly dominant diagonal and $0 < \varepsilon \neq 1$, then the states $\varrho(A, \varepsilon)$ are separable.

(2) If a positive semidefinite matrix A has a dominant diagonal maximally broken and $0 < \varepsilon \neq 1$, then the states $\varrho(A, \varepsilon)$ are entangled.

An interesting class of $\varrho(A, \varepsilon)$ is given by $A = \mathbf{1}$, where

$$\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is characterised by a circulant structure of the principal diagonal with entries

$$\{(1, \varepsilon, \varepsilon^{-1}), (\varepsilon^{-1}, 1, \varepsilon), (\varepsilon, \varepsilon^{-1}, 1)\}\,. \tag{7}$$

It was shown in [7] and in [2, p. 120] that the states $\varrho(\mathbf{1}, \varepsilon)$ are entangled if $0 < \varepsilon \neq 1$, which remains in the perfect agreement with Hypothesis 1. Note that $\mathbf{1}$ has a dominant diagonal maximally broken!

[2]Generalised realignment criterion CCNR can be formulated as follows: If the state of a composite system ρ_{AB} is separable, then for each $u \in [-1, 1]$ the quantity

$$z_u[\rho_{AB}] := |\mathcal{R}(\rho_{AB} + u\rho_A \otimes \rho_B)|_1 - \sqrt{(1 + u \operatorname{tr} \rho_A^2)(1 + u \operatorname{tr} \rho_B^2)} \leqslant 0\,,$$

where $\mathcal{R}$ is a realignment map and $|\cdot|_1$ is the trace norm. As a consequence, if the above condition is broken, then the state ρ_{AB} is entangled.

[3]A $d \times d$ matrix $X = [x_{ij}]$ is called

— *a dominant diagonal matrix*, if $\forall\, i = 1, \ldots, d \quad a_{ii} \geqslant \sum_{j, j \neq i} |a_{ij}|$,

— *a stricly dominant diagonal matrix*, if $\forall\, i = 1, \ldots, d \quad a_{ii} > \sum_{j, j \neq i} |a_{ij}|$,

— *the matrix with maximally broken dominant diagonal*, if $\forall\, i = 1, \ldots, d \quad a_{ii} < \sum_{j, j \neq i} |a_{ij}|$.

5.2 Local numerical range analysis

An elegant method of detection of entanglement in some bipartite state ρ in a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is to find an appropriate *entanglement witness*, i.e., a self-adjoint operator W, which being non-negative on product states,

$$\langle\psi| \otimes \langle\phi|W|\phi\rangle \otimes |\psi\rangle \geqslant 0\,, \quad |\phi\rangle \in \mathcal{H}_A, \quad |\psi\rangle \in \mathcal{H}_B\,, \tag{8}$$

detects ρ, i.e., $\mathrm{tr}(\rho W) < 0$. There are various methods of construction of witnesses (cf. [2, Sect. 3.2.8]). One of them uses a simple observation that for each entanglement witness W there exist $\lambda > 0$ and a positive definite operator P, such that

$$W = \lambda \mathbb{1} - P\,.$$

The values of λ correspond to the so-called *local numerical range of* P ($\mathrm{LNR}(P)$). By this notion we mean (cf. [2, Sect. 1.6]) an interval $\mathrm{LNR}(P) = [a_{\min}, a_{\max}]$ (contained in spectrum of P), which covers the values of P on the product states, i.e.,

$$\mathrm{LNR}(P) = \{\langle y| \otimes \langle x|P|x\rangle \otimes |y\rangle : |x| = |y| = 1,\ |x\rangle \in \mathcal{H}_A,\ |y\rangle \in \mathcal{H}_B\}\,. \tag{9}$$

The condition (8)

$$0 \leqslant \langle\psi| \otimes \langle\phi|W|\phi\rangle \otimes |\psi\rangle = \lambda - \langle\psi| \otimes \langle\phi|P|\phi\rangle \otimes |\psi\rangle$$

implies then

$$\lambda \geqslant \sup_{\substack{|\psi|=1\\|\phi|=1}} \{\langle\psi| \otimes \langle\phi|P|\phi\rangle \otimes |\psi\rangle\} = a_{\max}\,.$$

Hence, the knowledge of $\mathrm{LNR}(P)$ for some positive operator P enables one to construct a family of entanglement witnesses. The choice of P should reflect the state to be detected by the witness. In general, local numerical ranges of operators can be determined numerically using for example self-convergence (see [2, Sect. 1.6.1]).

However, it will be convenient to have an analytical solution to this problem, at least for a certain class of operators. Motivated by this I studied LNR for *circulant operators* acting on $\mathbb{C}^2 \otimes \mathbb{C}^d$ [2, Sect. 1.6.2]. Just like in the case of circular states already discussed, circulant operators correspond to a circular decomposition of the Hilbert space

$$\mathbb{C}^2 \otimes \mathbb{C}^d = \bigoplus_{k=1}^{d} \Sigma_k \tag{10}$$

into orthogonal two-dimensional subspaces

$$\begin{aligned}
\Sigma_1 &= \mathrm{span}\Big\{|e_1\rangle \otimes |f_1\rangle, |e_2\rangle \otimes |f_2\rangle\Big\}\,,\\
\Sigma_2 &= \mathrm{span}\Big\{|e_1\rangle \otimes |f_2\rangle, |e_2\rangle \otimes |f_3\rangle\Big\}\\
&\vdots\\
\Sigma_d &= \mathrm{span}\Big\{|e_1\rangle \otimes |f_d\rangle, |e_2\rangle \otimes |f_1\rangle\Big\}\,,
\end{aligned}$$

where $\{|e_i\rangle \otimes |f_k\rangle\}$, $i = 1, 2$, $k = 1, \ldots, d$ is a basis in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^d$. A self-adjoint operator $\mathcal{O}$ is called *circulant* with respect to a decomposition (10), if

$$\mathcal{O} = \mathcal{O}_1 \oplus \ldots \oplus \mathcal{O}_d\,, \tag{11}$$

where each operator $\mathcal{O}_k$ acts on the subspace Σ_k and it can be represented in the basis $\{|e_i\rangle \otimes |f_k\rangle\}$ by a complex-entries matrix $[a_{ij}^{(k)}]$ of dimension 2×2 as

$$\mathcal{O}_k \;=\; \sum_{i,j=1}^{2} a_{ij}^{(k)} |e_i\rangle\langle e_j| \otimes |f_{i+k}\rangle\langle f_{j+k}| \,, \tag{12}$$

where addition in subscripts is assumed to be modulo d.

The crucial results of that study are the following:

(1) in the case of the circulant operator acting on $\mathbb{C}^2 \otimes \mathbb{C}^d$ it is always possible to choose such an orthonormal product basis that the entries of the matrix representation of this operator are real. In addition, maximal and minimal values of the function

$$f_{\mathcal{O}}(|x\rangle, |y\rangle) = \langle x| \otimes \langle y|\mathcal{O}|x\rangle \otimes |y\rangle$$

which determine $\mathrm{LNR}(\mathcal{O})$, are attained in points with real coordinates [2, Th. 1.2]. This allows one to reduce effectively the optimization problem from $\mathbb{C}^2 \otimes \mathbb{C}^d$ to $\mathbb{R}^2 \otimes \mathbb{R}^d$. However, because of a polynomial character of a set of $d + 2$ equations, it is rather hopeless to expect analytical solutions for $d > 2$,

(2) for $d = 2$, analytical solutions leading to $\mathrm{LNR}(\mathcal{O})$ are presented in [2, Sect. 1.6.3] and [8].

5.3 Analysis of SPPT states

An interesting and very important class of states is distinguished by the condition that the *partial transposition* acting on their matrix representations, i.e. the transposition only in one of the subsystems, does not destroy the semi-positivity of their spectrum. Such states are called PPT (positive partial transposed states). In $d_A \otimes d_B$ systems, for $d_A d_B \leqslant 6$, all PPT states are separable. In higher dimensions among PPT states there are also entangled states, but they are very weakly entangled, and detecting (and measuring) entanglement in that case is particularly difficult. It is due to the fact that the most efficient entanglement criterion based on partial transposition is invalid in this case (see [2, Sect. 3.1.4] as well as [2, Sect. 3.3.2]). Therefore, from a theoretical point of view, it is vital to provide a simple method of generation of such PPT states, in order to study their entanglement, and to distinguish some families of PPT states, which remain separable for any dimension of the Hilbert space.

Such studies were presented in [5, 6], where the following method of construction of new families of $d_A \otimes d_B$-PPT states, called SPPT (Strong PPT) and SSPPT (Super Strong PPT), was proposed:

(1) Let $\{|e_1\rangle, \ldots, |e_{d_A}\rangle\}$ be a basis in $\mathbb{C}^{d_A}$.

(2) Given matrices X_i and S_{ij}, $i < j$, $i, j = 1, \ldots, d_A$ of dimension $d_B \times d_B$ consider a block matrix

$$\boldsymbol{X} = \left[\begin{array}{c|c|c|c|c} X_1 & S_{12}X_1 & S_{13}X_1 & \cdots & S_{1d_A}X_1 \\ \hline 0 & X_2 & S_{23}X_2 & \cdots & S_{2d_A}X_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & X_{d_A-1} & S_{d_A-1,d_A}X_{d_A-1} \\ \hline 0 & 0 & 0 & 0 & X_{d_A} \end{array}\right].$$

(3) A normalised matrix $\frac{1}{N}\boldsymbol{X}^\dagger\boldsymbol{X} =: \rho_{AB}$, where $N = \mathrm{tr}(X_1) + \ldots + \mathrm{tr}(X_{d_A})$, is called a *SPPT state*, if there exists another block matrix

$$\boldsymbol{Y} = \left[\begin{array}{c|c|c|c|c} X_1 & S_{12}^\dagger X_1 & S_{13}^\dagger X_1 & \cdots & S_{1d_A}^\dagger X_1 \\ \hline 0 & X_2 & S_{23}^\dagger X_2 & \cdots & S_{2d_A}^\dagger X_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & X_{d_A-1} & S_{d_A-1,d_A}^\dagger X_{d_A-1} \\ \hline 0 & 0 & 0 & 0 & X_{d_A} \end{array}\right],$$

such that $\rho_{AB}^{T_A} = \frac{1}{N}\boldsymbol{Y}^\dagger\boldsymbol{Y}$.

(4) In that case we declare that a $d_A \otimes d_B$-state ρ_{AB} has a *SPPT representation* (from the side B) corresponding to X_i and S_{ij}.

In the analogue way one can define the SPPT representation from the side A. Certainly, the method of construction guarantees that the states are PPT.

From the condition $\rho_{AB}^{T_A} = \frac{1}{N}\boldsymbol{Y}^\dagger\boldsymbol{Y}$ it results the following compatibility conditions between X_i and S_{ij}:

$$\sum_{k=1}^{j-1} X_k^\dagger S_{kj}^\dagger S_{kj} X_k = \sum_{k=1}^{j-1} X_k^\dagger S_{kj} S_{kj}^\dagger X_k \quad \text{dla} \quad j = 2, \ldots, d_A\,, \tag{13}$$

$$\sum_{k=1}^{i-1} X_k^\dagger S_{kj}^\dagger S_{ki} X_k = \sum_{k=1}^{i-1} X_k^\dagger S_{ki} S_{kj}^\dagger X_k \quad \text{dla} \quad 2 \leqslant i < j = 3, \ldots, d_A\,, \tag{14}$$

In particular, (13) and (14) are fulfilled, if for all $k < i \leqslant j$

$$S_{ki} S_{kj}^\dagger = S_{kj}^\dagger S_{ki}\,. \tag{15}$$

States which have the SPPT representation corresponding to the family of matrices $\{S_{ik}\}$ fulfilling (15) are called *SSPPT* (Super Strong PPT).

Several properties of these states, in particular, their entanglement/separability property, representability of a given state in a SPPT (SSPPT) form, some connections of the SSPPT states with the zero-discord states have been studied in [5, 6] as well as in [2, Sect. 3.3.3]. The separability problem of SPPT states can be summarized as follows:

(1) SSPPT states in $d_A \otimes d_B$ systems are separable [2, Thm. 3.22]. Hence, to represent SPPT state using the family of *normal* matrices $\{S_{ik}\}$ satisfying conditions (15) is enough to decompose it as a convex combination of product states of subsystems. Moreover, this decomposition is in some situations unique [2, Thm. 3.23].

(2) If $d \leqslant 4$, then the SPPT states in $2 \otimes d$ systems are separable [2, Thm. 3.24]. For $d \geqslant 5$ there are entangled SPPT states!

Another problem is raising: can a given $d_A \otimes d_B$-state ρ_{AB} be represented as SPPT state? In the case of $2 \otimes d$-states, analysed in [2, Ch. 3.3.3], it is equivalent to the problem of existing X_1, X_2 and S fulfilling the condition

$$X_1^\dagger S^\dagger S X_1 = X_1^\dagger S S^\dagger X_1\,. \tag{16}$$

The following two simple cases do satisfy this condition:

(1) If S is self-adjoint, then the condition (16) is fulfilled, moreover, such a state is PPT-invariant, i.e. $\boldsymbol{X} = \boldsymbol{Y}$, as a consequence, $\rho_{AB} = \rho_{AB}^{T_A}$.

(2) If X_1 has a maximal rank (equal to d), then the condition (16) is fulfilled if and only if the matrix S is normal, i.e. $[S, S^\dagger] = 0$. Note that for $2 \otimes d$ systems every state of rank at least d can be described in a canonical form

$$\rho_{AB} = \left[\begin{array}{c|c} \mathbb{1} & S \\ \hline S^\dagger & S^\dagger S \end{array}\right]. \tag{17}$$

It is obvious that the above states are SPPT, for which $X_1 = \mathbb{1}$ and $X_2 = 0$.

In both cases, the resulting SPPT states are separable, as states of the $2 \otimes d$ system of rank at least d.

More general situation occurs when in the given basis the matrix representation of the state does not display its canonical form (17) but

$$\rho_{AB} = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{21} & \rho_{22} \end{array}\right], \tag{18}$$

where the block ρ_{11} has maximal rank. Then the SPPT representation exists if and only if [2, Thm. 3.19]

$$\rho_{12}^\dagger \rho_{11}^{-1} \rho_{12} = \rho_{12} \rho_{11}^{-1} \rho_{12}^\dagger . \tag{19}$$

When $\mathrm{rank}(\rho_{11}) = r < d$ the situation is more complicated, but it is still possible to represent the state in the SPPT form, if the following condition is fulfilled [2, Thm. 3.20]

$$s_{11}^\dagger s_{11} + s_{21}^\dagger s_{21} = s_{11} s_{11}^\dagger + s_{12} s_{12}^\dagger , \tag{20}$$

where s_{ik} are such blocks of S, that the dimension of s_{11} is $r \times r$,

$$S = \left[\begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array}\right].$$

Interestingly, it turned out that the classical-quantum and quantum-classical states (3) in $2 \otimes d$ systems (but not in $3 \otimes d$) may be represented as SSPPT states. It was first observed in [10], and later discussed, in a slightly different context, in [2, Thm. 3.21].

5.4 Estimations of concurrence using entanglement witnesses

It is a serious disadvantage of many entanglement measures that they can be easily calculated only for pure states and for a few families of states with high symmetry (Werner, isotropic, orthogonally invariant states). Therefore, it is crucial to look for computable estimates for various entanglement measures and for their relationships among them.

The most fundamental meaning as a measure of correlations is a information-theoretic quantity called *entanglement entropy*. In the case of $2 \otimes 2$ systems it is a function of *concurrence*, which is another quantity used to evaluate the degree of entanglement, however, is not strictly an entanglement measure (it does not reduce to the entanglement entropy for pure states). Note that various measures and quantities related to them enable often a quantitative evaluation of other aspects of entanglement corresponding to various strategies of its use.

Let $|\psi_{AB}\rangle$ be a pure state in $d \otimes d$ system. *Concurrence* of the pure state is defined by

$$\mathcal{C}(|\psi_{AB}\rangle) = \sqrt{2(1 - \mathrm{tr}\, \rho_A^2)}\,.$$

Simple calculation shows that concurrence is a function of Schmidt coefficients $\{\sqrt{\mu_k}\}$ of $|\psi_{AB}\rangle$:

$$\mathcal{C}(|\psi_{AB}\rangle) = 2\sqrt{\sum_{i<j}\mu_i\mu_j}\,. \tag{21}$$

Its generalisation to mixed states is obtained via the convex roof construction (see [2, Def. 3.15]):

$$C(\rho) = \operatorname{co}\mathcal{C}(\rho) = \inf_{\{p_k,|\psi_k\rangle\}}\Big\{\sum_k p_k\mathcal{C}(|\psi_k\rangle)\ :\ \rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|\Big\}\,. \tag{22}$$

Concurrence for two-qubit mixed states was first derived by Wooters as

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}\,, \tag{23}$$

where λ_k are singular values of non-hermitian matrix $\rho\tilde{\rho}$ in non-increasing order, $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ is a matrix corresponding to spin flip. The relation (23) does not generalise to higher dimensions $d > 2$, but some similar relations can be obtained as lower estimations. In [2, Ch. 3.5.1] I called this type of estimation an *algebraic* one but there are also other methods of estimation using various norms (trace-norm, cross-norm, and others) (see [2, Sect. 3.5.2]), entanglement witnesses or positive maps. In particular, the last two methods were analysed in [11, 12] and in [2, Ch. 3.5.3,Ch. 3.5.4].

A possibility of concurrence estimation using entanglement witnesses results from the following observation [2, Thm. 3.29]: let $f[\rho]$ be a convex functional on the space of a $d \times d$ composite system fulfilling the condition

$$f[|\psi\rangle\langle\psi|] \ \leqslant\ 2\sum_{k<l}\sqrt{\mu_k\mu_l} \tag{24}$$

for every pure state with Schmidt coefficients $\{\sqrt{\mu_k}\}$, then

$$C(\rho) \ \geqslant\ \sqrt{\frac{2}{d(d-1)}}\,\max\{0, f[\rho]\}\,.$$

Note, that the functional f can be chosen as

$$f[\rho] = -\operatorname{tr}(\rho W)\,,$$

where W denotes an entanglement witness of ρ. Note that providing the condition (24) is valid, i.e.

$$-\langle\psi|W|\psi\rangle \leqslant 2\sum_{k<l}\sqrt{\mu_k\mu_l}\,, \tag{25}$$

the witness W of ρ gives the following estimation of concurrence:

$$C(\rho) \geqslant \sqrt{\frac{2}{d(d-1)}}|\operatorname{tr}(\rho W)|\,. \tag{26}$$

It seems that the condition (25) distinguishes a class of witnesses obeying it, but in fact, every witness detecting ρ after an appropriate scaling does fulfil it. Indeed, a witness multiplied by a positive factor is still a witness and, as a consequence, can be used to estimate concurrence [11, 12]. Note also that for the pure state with the Schmidt decomposition

$$|\psi\rangle = \sum_{k=1}^{d}\sqrt{\mu_k}|a_k\rangle \otimes |b_k\rangle \tag{27}$$

the expectation value of W in the state ρ reads

$$\langle\psi|W|\psi\rangle = \sum_{k,l} \sqrt{\mu_k \mu_l} A^{(W)}_{kl}(\psi)\,,$$

where the matrix elements of $A^{(W)}(\psi)$ are the following:

$$A^{(W)}_{kl}(\psi) = \mathrm{Re}\,\langle a_k| \otimes \langle b_k|W|a_l\rangle \otimes |b_l\rangle\,. \tag{28}$$

In particular, diagonal entries $A^{(W)}_{kk} = \langle a_k| \otimes \langle b_k|W|a_k\rangle \otimes |b_k\rangle$ are expectation values of W on separable states, therefore,

$$A^{(W)}_{kk}(\psi) \geqslant 0\,. \tag{29}$$

In this new notation the condition (25) is equivalent to

$$\sum_{k,l} \sqrt{\mu_k \mu_l}(A^{(W)}_{kl}(\psi) + 1) \;\geqslant\; 1\,. \tag{30}$$

Now, define a quantity

$$\lambda(W) := -\min_{\psi} \min_{k\neq l} A^{(W)}_{kl}(\psi)\,, \tag{31}$$

which gives a new characterisation of an entanglement witness in terms of matrix elements of $A^{(W)}(\psi)$. In fact, the quantities defined in (28) are, for a given witness W, polynomials in $(\boldsymbol{x}, \boldsymbol{y}) = (x^{(k)}_j, y^{(l)}_s)$: coordinates of Schmidt vectors $|a_k\rangle = [x^{(k)}_1, \ldots, x^{(k)}_d]$, $|b_l\rangle = [y^{(l)}_1, \ldots, y^{(l)}_d]$. In that picture, $-\lambda(W)$ is an absolute minimum of the whole family of polynomials corresponding to off-diagonal entries of (28). If

$$\min_{\psi} \min_{k\neq l} A^{(W)}_{kl}(\psi) \geqslant 0\,,$$

then W could not detect any entangled state, hence the absolute minimum should be negative and hence $\lambda(W) > 0$. Now, for every witness W detecting ρ, a rescaled witness $W_\alpha = \alpha^{-1} W$ fulfils (30), providing that $\alpha \geqslant \lambda(W)$. As a consequence W_α can be used to estimate concurrence via

$$C(\rho) \;\geqslant\; \sqrt{\frac{2}{d(d-1)}}\,|\,\mathrm{tr}(\rho W_\alpha)|\,. \tag{32}$$

To summarize, I proposed the following procedure which can be applied to estimate concurrence of ρ using its entanglement witness W:

$$W \;\xrightarrow{(1)}\; W' \;\xrightarrow{(2)}\; A^{(W')}_{kl} \;\xrightarrow{(3)}\; \lambda(W') \;\xrightarrow{(4)}\; W_\lambda \;\xrightarrow{(5)}\; \mathrm{tr}(W_\lambda \rho)\,.$$

The subsequent steps have the following meaning:

(1) determining $\kappa = \min_k A^{(W)}_{kk}$. If $\kappa > 0$, then one can construct the better (tangent) witness, i.e., $W' = W - \kappa \mathbb{1}$,

(2) determining the family of polynomials $\{W_{ij}(\boldsymbol{x}, \boldsymbol{y})\}$ corresponding to off-diagonal entries of $A^{(W')}(\psi)$,

(3) calculating an absolute minimum $\lambda(W')$ of a set of polynomials $\{W_{ij}(\boldsymbol{x}, \boldsymbol{y})\}$,

(4) determining the rescaled witness $W_\lambda = \lambda^{-1} W'$,

(5) estimating concurrence.

The estimation using entanglement witnesses has two advantages compared to those obtained by other methods,

(1) if we know an entanglement witness of some state, it is much simpler than in other methods raising from optimisation procedures to determine the estimation of concurrence. Unfortunately, the value of $\lambda(W)$ must be often determined numerically,

(2) in principle, the estimation using witnesses is directly measurable in experiment, as an average value of the self-adjoint operator in a quantum state.

Going further, the concurrence estimation provided by entanglement witnesses can be pulled forward to an estimation introduced directly by positive maps (see [2, Ch. 3.5.4]).[4] Positive but not completely positive maps (n-CP) play the crucial role in entanglement detection (see [2, Ch. 3.2.7]) and in constructions of entanglement witnesses using Choi-Jamiołkowski isomorphism $\mathcal{J} : \Phi \mapsto W_\Phi$ defined as

$$W_\Phi = \mathcal{J}(\Phi) := \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)\,,$$

where $\{|i\rangle\}$ is a computational basis in $\mathbb{C}^d$.[5] Suppose that Schmidt bases (27) $\{|a_k\rangle\}$, $\{|b_k\rangle\}$ corresponding to the pure state $|\psi\rangle$ have entries

$$|a_k\rangle = [a_1^{(k)}, \ldots, a_d^{(k)}]\,, \qquad |b_k\rangle = [b_1^{(k)}, \ldots, b_d^{(k)}]\,.$$

Then, for $k \neq l$ the quantities $A_{kl}^{(\Phi)}(\psi)$, which are analogues of (28), but corresponding to Φ, read

$$\begin{aligned} A_{kl}^{(\Phi)}(|a_k\rangle, |b_l\rangle) &= \sum_{i,j} \mathrm{Re}\Big[\langle a_k|i\rangle\langle j|a_l\rangle\langle b_k|\Phi(|i\rangle\langle j|)|b_l\rangle\Big] \\ &= \sum_{i,j}\sum_{p,q} \mathrm{Re}\Big[\Phi_{pq}^{ij} a_i^{(k)} a_j^{(l)*} b_p^{(k)} b_q^{(l)*}\Big]\,, \end{aligned}$$

where

$$\Phi_{pq}^{ij} = \langle p|\Phi(|i\rangle\langle j|)|q\rangle\,. \tag{33}$$

The concurrence estimation resulting directly from (32), takes the form

$$C(\rho) \geqslant \sqrt{\frac{2}{d(d-1)}}(\lambda(\Phi))^{-1} |\,\mathrm{tr}(\rho(\mathbb{1}\otimes\Phi)(\Pi_d^+))|\,, \tag{34}$$

[4] Let us recall that a map $\Phi : M_d \to M_d$, where M_d denotes the set of $d \times d$ matrices is called

— *positive*, if it preserves hermicity and positivity semidefinitness of matrices,

— *k-positive*, if the mapping $\mathbb{1}_k \otimes \Phi : M_k \otimes M_d \to M_k \otimes M_d$ is positive,

— *completely positive* (CP), if the mapping $\mathbb{1}_k \otimes \Phi : M_k \otimes M_d \to M_k \otimes M_d$ is positive for $k \leqslant d$,

— *decomposable*, if there exist completely positive maps Φ_1 and Φ_2, such that $\Phi = \Phi_1 + \Phi_2 \circ T$.

[5] Isomorphism $\mathcal{J}$ gives a one-to-one correspondence between the set of linear maps $\{\Phi : M_d \to M_d\}$ and the set of entanglement witnesses $\{W_\Phi\}$ displaying the following properties [2, Ch. 1.5.2]:

(1) Φ is a CP map iff W_Φ is positive semi-definite.

(2) Φ is a n-CP map iff W_Φ is a block-positive matrix, which is not positive, i.e.,

$$\langle x| \otimes \langle y|W_\Phi|x\rangle \otimes |y\rangle \geqslant 0\,, \quad \forall\, |x\rangle, |y\rangle\,.$$

(3) Φ is decomposable iff W_Φ is decomposable, i.e., $W_\Phi = W_1 + W_2^{T_B}$ for some positive semi-definite operators W_1, W_2.

where

$$-\lambda(\Phi) = \min_{|a_k\rangle,|b_l\rangle} \min_{k\neq l} A_{kl}^{(\Phi)}(|a_k\rangle, |b_l\rangle)$$

Unfortunately in most known examples, the quantity $\lambda(\Phi)$ must be determined numerically due to a large number of variables to be optimised upon.

5.5 Analysis of quantum discord related to the Tsallis entropy function

The definition of quantum discord (5) can be generalised using information-theoretic functions other than von Neumann entropy. They can be used to determine classical and total correlations in the system (see for example a discussion in [2, Ch. 1.4]). In particular, Tsallis entropy function

$$T_q(X) = \frac{1 - \operatorname{tr} \rho_X^q}{q-1}\,, \quad q > 0\,, \quad q \neq 1\,, \quad X = A, B, AB \tag{35}$$

or Renyi entropy

$$R_q(X) = \frac{\ln(\operatorname{tr} \rho_X^q)}{q-1}\,, \quad q > 0\,, \quad q \neq 1\,, \quad X = A, B, AB\,,$$

where ρ_{AB}, ρ_A, ρ_B denote the state of the composite system AB, and its reductions to subsystems A and B, respectively, seem to be good candidates.[6] Both functions generalising von Neumann entropy have been successfully used in entanglement detection procedures (extensive discussion and summary of this topic can be found in [2, Sect. 3.2.3]) giving some new insight into of quantitative description of correlations different than von Neumann entropy.

Hopefully, a similar situation can also be true when examining quantum discord. Therefore, I introduced a notion of a *quantum q-discord* based on Tsallis entropy in [13, 14, 15] and in [2, Rozdz. 4.4].

Quantum q-discord (from a side A) is defined as (see (5))

$$D_{qAB} := I_{qAB} - C_{qAB}\,, \tag{36}$$

where

$$I_{qAB} = T_q(A) - \frac{T_q(B) - T_q(AB)}{1 + (1-q)T_q(B)} = T_q(A) - T_q(A|B)$$

is a mutual Tsallis entropy (a counterpart of mutual von Neumann entropy interpreted as a measure of total correlations in ρ_{AB}), and

$$C_{qAB} = T_q(A) - \inf_{\{\Pi^B\}} T_q(A|\{\Pi^B\})$$

is a mutual Tsallis entropy conditioned by a measurement Π^B (a counterpart of von Neumann entropy conditioned by a measurement interpreted as a measure of classical correlations in ρ_{AB}). The quantity

$$T_q(A|\{\Pi^B\}) = \frac{\sum_k p_k^q T_q(\rho_k^A)}{\sum_k p_k^q} \tag{37}$$

is a Tsallis entropy of subsystem A conditioned by a measurement on subsystem B represented by an ensemble of post-measurement states $\{p_k, \rho_k^A\}$. It is relatively easy to check that the relation

$$D_{qAB} = \inf_{\{\Pi^B\}} T_q(A|\{\Pi^B\}) - T_q(A|B)\,, \tag{38}$$

[6] Both functions T_q and R_q give the von Neumann entropy H in the limit $q \to 1$.

describes the q-discord as well. The first term in (38) needs optimisation over measurements on subsystem B, but the second being a conditional Tsallis entropy is measurement-independent. Note two fundamental differences of new-defined quantities with respect to ordinary discord:

(1) in the definition of q-discord a one-parameter family of Tsallis entropy (35) indexed by $q > 0$ is used. This allows to observe various types of behaviour of quantum correlations with respect to q,

(2) the relation (37) uses a modified procedure of averaging over the ensemble (generalised Holevo quantity) (detailed discussion can be found in [2, Ch. 1.4]).

In [2, Sect. 4.4] I derived q-discord for two-qubits Werner and isotropic states showing analytically that for these highly symmetric states it is non-negative and it takes zero only for a maximally mixed state, as opposed to other measures based on Tsallis entropy. It should be emphasized that q-discord allows a selection of some correlation features depending on the value of a continuous parameter q. Unfortunately, the values of q-discord are not monotonic with respect to q even for symmetric states. Moreover, analysis of a q-discord for some particular family of circulant states shows that it can take negative values for some $q \geqslant 2$, which significantly limits its use as a quantum correlation measure, however, it does not exclude it completely. Numerical analysis shows that q-discord is still non-negative for a wide class of states when $0 < q \leqslant 1$.

To summarize the results of my research in the field of quantum correlations, their detection and characterization let me emphasise the following:

(1) we have examined a subclass of $3 \otimes 3$ circulant states. We have shown that some of them are entangled (states $\varrho(\mathbf{1}, \epsilon)$) and as to the others we have postulated a hypothesis about the relationship between separability/entanglement and the notion of a dominant diagonal of the matrix,

(2) we have shown that every circulant operator acting on $\mathbb{C}^2 \otimes \mathbb{C}^d$ can be represented (in a suitable basis) by a real-entries matrix, moreover, in this basis, supremum and infimum of a local numerical range (LNR) are attained on vectors with real components. For $d = 2$, we provided their analytical forms, hence giving the LNR,

(3) we defined and analysed new families of PPT states called SPPT and SSPPT, which obey a chain of incusions

$$\text{SSPPT} \ \subset \ \text{SPPT} \ \subset \ \text{PPT}\,.$$

We have shown that SSPPT $\subset$ SEP and SPPT($d \leqslant 4$) $\subset$ SEP,[7]

(4) we have derived conditions under which any state of a composite $2 \otimes d$ system has a SPPT representation (from the point of view of one of the subsystems). In particular, we have shown that all classical-quantum and quantum-classical states of $2 \otimes d$ system (but not $3 \otimes d$) can be represented in the SSPPT form,

(5) we have discussed a method of estimating concurrence based on entanglement witnesses. We have shown that each witness detecting entanglement of the given state can be

[7] SEP denotes here the set of separable states whereas by SPPT($d \leqslant 4$) we mean the set of $2 \otimes d$ SPPT states with $d \leqslant 4$.

appropriately rescaled and used to estimate concurrence. The optimal value of this rescaling $\lambda(W)$ is a new parameter characterizing the witness. A similar notion can be defined for n-CP mappings (positive but not completely positive),

(6) we have introduced a new measure of quantum correlations — quantum discord based on Tsallis entropy (q-discord). We have shown that it is well-defined (non-negative) for highly symmetric two-qubit states (Werner and isotropic). We have determined its value for a class of circulant states. We discussed and compared the q-discord with other similar measures of correlations,

(7) in my monograph [2] one finds many analyses of quantities describing correlations for symmetric (Werner, isotropic, orthogonal) and circulant states as well as discussions of entropic quantities used to characterise correlations, including quantum discord and q-discord.

6 Other scientific achievements

6.1 Applications of Lie-algebraic methods for solving partial differential equations

My earlier (pre-doctoral) interest was also related to the subject of spectral line shapes, in particular, to the influence of the measuring apparatus on the line shape [16, 17]. It turned out that the line profile $I(\omega)$ can be determined from solutions of some partial differential equations. Providing some additional assumptions these solutions can be obtained analytically [18] using the Lie-algebraic methods in application to the set of differential operators (see for instance [19]).

Width, shift with respect to the unperturbed frequency and possible asymmetry of the line are closely related to the conditions under which a line arises: they depend on temperature and pressure of radiating gas, as well as on the presence of other gases, which can collide with emitters and change their movement.

In the simplest approximation, if one neglects any influence of collisions on emitting atoms, the line profile (the so-called Voigt profile) is symmetric and takes into account the Doppler width and shift (corresponding to the movement of emitters) and pressure width and shift (corresponding to pressure of radiating gas). Experimental line profile is typically a convolution of a Voigt profile with an instrument function that characterises influence of the measuring apparatus (for instance, of interferrometer used in the experiment) on line parameters.

In more advanced models one should take into account the impact of collisions on the line shape, in particular, the influence of speed changing collisions with perturbing atoms on the shift and width of the line. The line profile can be determined from a distribution function $F(t,\vec{v})$, depending on time and a velocity vector $\vec{v}$, which satisfies the following Boltzmann kinetic equation:

$$\frac{\partial}{\partial t}F(t,\vec{v}) = -i(\omega_0 + \vec{k}\cdot\vec{v})F(t,\vec{v}) + \widehat{S}F(t,\vec{v})\,,$$

where ω_0 is the rest transition frequency and $\vec{k}$ is the wave vector of emitting light. The collision term $\widehat{S}F(t,\vec{v})$ includes all collisional corrections. In the soft collision model and assuming the quadratic dependence of collisional width and shift on the emitter speed, the line shape can be finally determined by solving the differential equation

$$\frac{\partial}{\partial t}G(t,x) = \widehat{A}G(t,x)\,, \qquad G(0,x) = e^{-x^2}\,, \tag{39}$$

with the differential operator

$$\widehat{A} = c_1 + c_2 x + c_3 x^2 + c_4 x \frac{\partial}{\partial x} + c_5 \frac{\partial^2}{\partial x^2}\,, \tag{40}$$

where c_k are some complex constants.

From the Lie-algebraic point of view the differential operator (40) belongs to the algebra spanned by [21]

$$\left\{\hat{e}_1 = 1,\ \hat{e}_2 = x,\ \hat{e}_3 = x^2,\ \hat{e}_4 = x\frac{\partial}{\partial x},\ \hat{e}_5 = \frac{\partial^2}{\partial x^2},\ \hat{e}_6 = \frac{\partial}{\partial x}\right\},$$

hence, the solution to (39) belongs to the orbit of an appropriate Lie group passing through an initial point $G(0, x)$, i.e.,

$$G(t, x) = e^{t\hat{A}} G(0, x)\,. \tag{41}$$

Methods of describing the orbits of Lie groups are well-known (see for instance, [19, 21]). What significantly reduces (or even enables) analytical results is the initial condition $G(0, x)$ in the form of a Gauss function, i.e., a function belonging to a family

$$\{e^{-(x-\mu)^2/\sigma^2} : \sigma^2 > 0,\ \mu \in \mathbb{R}\}\,.$$

It is obvious due to the Baker-Hausdorff formula that there exist functions $g_k(t)$, $k = 1, \ldots, 6$, which allow to display the orbit of the group (41) as a composition

$$G(t, x) = \exp[g_1(t) + g_2(t)x + g_3(t)x^2] \exp\left[g_4(t) x \frac{\partial}{\partial x}\right] \times \exp\left[g_5(t) x \frac{\partial^2}{\partial x^2}\right] \exp\left[g_6(t) \frac{\partial}{\partial x}\right] e^{-x^2}, \tag{42}$$

where each operator $\exp[g_k(t)\hat{e}_k]$ transforms a Gauss function into another (for details see [18]). Hence the solution can be written as

$$G(t, x) = \exp[h_1(t) + h_2(t)x + h_3(t)x^2]\,,$$

where the functions $h_k(t)$ obey the set of ordinary differential equations which can be solved analytically [18]! Now, in soft collision approximation and assuming the quadratic dependence of collisional width and shift on velocity, $F(t, \vec{v})$ can be determined from $G(t, x)$, and finally $I(\omega)$ can be obtained via

$$I(\omega) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{i\omega t} \Phi(t) dt\,, \qquad \Phi(t) = \int_{\mathbb{R}^3} F(t, \vec{v}) d\vec{v}\,.$$

To summarize my achievements:

(1) using Lie-algebraic methods in application to a particular set of differential operators we have solved a class of Boltzmann kinetic equations with collisional operator of the Fokker-Planck-type taking into account the velocity-changing collisions and the particular quadratic dependence of the collisional line width and shift on velocity,

(2) we have discussed the impact of such effects on the line-shape profile and we have compared them with known results.

6.2 Quantisation of dissipative systems

The quantum description of physical system which dissipate energy during evolution (eg., as a result of various types of damping forces) is still far away from full understanding. Such non-Hamiltonian systems cannot be, in general, quantise using canonical methods. In principle, there is no room for their description in the standard formulation of quantum mechanics based on a Hilbert space formalism and the Schrödinger equation which provides a one-parameter family of unitary operators defining reversible evolution. Beyond this formalism is a quantum theory of open systems permitting non-unitary evolution (based on semigroups formalism) with dissipation [20].

There are, however, other methods involving no such a general theory. There are deformation quantisation raised from the classical phase space with noncommutative composition (convolution) of functions or the methods based on direct quantisation of classical equation of motion, just to recall a few. Unfortunately, apart from some simplest cases, these methods are difficult to generalize and to apply.

In what follows I would like to focus on discussing different methods of canonical quantisation of the damped harmonic oscillator (DHO) model described by a classical differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0\,, \tag{43}$$

where γ is the damping constant, ω_0 is the eigen-frequency. It is one of the simplest, yet most important models, which has been extensively studied and can serve as a toy model for testing various methods (a short discussion and the subject and further references can be found in [22]). The questions which arise are related to the following problems:

— how to represent this equation in order to derive its quantum counterpart,

— how to solve the quantum equation,

— and finally, how to interpret the results, in particular, what is and how can we recognise dissipation at the quantum level?

All above questions seem to be non-trivial!

In 1931 Bateman [23] tried to represent DHO as a Hamiltonian system with an additional degree of freedom y, which would obey a "dual" equation with an opposite sign of the dissipative term:

$$\ddot{y} - 2\gamma\dot{y} + \omega_0^2 y = 0\,. \tag{44}$$

The parameter y plays the role of a variable corresponding to a reservoir degree of freedom and it is responsible for receiving dissipated energy, so that the whole system (DHO + reservoir) is isolated. Constant of motion (the so-called *Bateman Hamiltonian*) of the dynamical system (43) + (44) takes the form

$$H(x, y, p_x, p_y) = p_x p_y + \omega^2 xy - \gamma(xp_x - yp_y)\,, \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}\,, \tag{45}$$

where canonical momenta are $p_x = \dot{y} - \gamma y$, $p_y = \dot{x} + \gamma x$. Unfortunately, this Hamiltonian does not reproduce the energy of the system even in the limit $\gamma \to 0$! Meanwhile using (45) Feshbach and Tichochinsky [24] have carried out the canonical quantisation procedure using a group SU(1,1) and its generators providing energies and energy eigenstates. The eigenvalues of the quantum counterpart of (45) indexed by two numbers

$$j = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots, \qquad m = |j|, |j|+1, |j|+2, \ldots\,,$$

turn out to be complex

$$E^{\pm}_{jm} = 2\hbar\omega j \pm i\hbar\gamma(2m+1)\,, \tag{46}$$

and the corresponding eigenstates — unnormalisable in a Hilbert space. Let us emphasize, however, that the Hamilton operator considered by Feshbach and Tichochinsky is self-adjoint. Numerous interpretations and generalizations of the results were reported by many authors (extensive discussion of these issues can be found in [25]).

In paper [26] we discussed a different method of quantisation of DHO system showing, among others, that complex energy values (46) of a self-adjoint Hamiltonian correspond to the so-called *resonant states*, well-known from the theory of scattering. Our approach used a different extension (providing additional degrees of freedom) of the classical phase space to obtain a Hamiltonian system. Note that due to Pontryagin [27] any dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x})\,, \qquad \boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N \tag{47}$$

where $\boldsymbol{X}$ is a vector field on a configuration space $\mathbb{R}^N$ can be generalised to a Hamiltonian system on $\mathbb{R}^N \times \mathbb{R}^N$ with

$$H(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{X}(\boldsymbol{x}) := \sum_{\ell=1}^{N} p_\ell X_\ell(\boldsymbol{x})\,. \tag{48}$$

Note that a half of Hamilton equations, i.e.,

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{p}} = \boldsymbol{X}(\boldsymbol{x})$$

reproduces the dynamical system under consideration (47), while the rest describes the evolution of additional degrees of freedom

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}}\,.$$

Quantisation of such system relies on a Wigner-Weyl (WW) transform of a classical Hamilton function (48), i.e., $\widehat{H}_{\text{quant}} = \text{WW}(H)$ (see [26] for details). Note that $\widehat{H}_{\text{quant}}$ is a self-adjoint operator on $L^2(\mathbb{R}^N, d\boldsymbol{x})$.

There is one more ambiguity in an application of this method to DHO system: the choice how to represent a second-order differential equation as a set of first order equations. We assumed the set of equations on $\mathbb{R}^2$ in the following form:

$$\begin{cases} \dot{x}_1 &= -\gamma x_1 + \omega x_2 \\ \dot{x}_2 &= -\gamma x_2 - \omega x_1\,. \end{cases}$$

The above set is non-Hamiltonian, but using Pontriagin method can be extend to a Hamiltonian one on $\mathbb{R}^2 \times \mathbb{R}^2$ with the following Hamiltonian function

$$H(\boldsymbol{x}, \boldsymbol{p}) = \omega(p_1 x_2 - p_2 x_1) - \gamma(p_1 x_1 + p_2 x_2)\,. \tag{49}$$

Now the variables (x_1, x_2, p_1, p_2) are in a very simple manner connected with the variables of a Bateman approach (45)

$$\begin{aligned} x_1 &= \frac{p_y}{\sqrt{\omega}}\,, & p_1 &= -\sqrt{\omega}\, y \\ x_2 &= -\sqrt{\omega}\, x\,, & p_2 &= -\frac{p_x}{\sqrt{\omega}}\,. \end{aligned}$$

It is convenient to carry out quantisation procedure in polar coordinates

$$x_1 + ix_2 = re^{i\varphi}$$

representing quantum Hamiltonian as

$$\widehat{H}_{\text{quant}} = i\omega\hbar\frac{\partial}{\partial\varphi} + i\gamma\hbar\Big(r\frac{\partial}{\partial r} + 1\Big)$$

on the Hilbert space $L^2(\mathbb{R}^2, dx_1dx_2) = L^2([0,2\pi), d\varphi)\otimes L^2(\mathbb{R}_+, rdr)$. Solving the eigenproblem $\widehat{H}_{\text{quant}}\Psi_{l\lambda} = E_{l\lambda}\Psi_{l\lambda}$, we obtained the energy eigenvalues [26, §4.2]

$$E_{l\lambda} = \hbar(l\omega + \lambda\gamma) \tag{50}$$

and eigenstates:

$$\Psi_{l\lambda}(r,\varphi) = \Phi_l(\varphi)\cdot R_\lambda(r) = \frac{1}{2\pi}r^{-(i\lambda+1)}e^{-il\varphi},$$

where $l = 0, \pm1, \pm2, \dots$ and $\lambda \in \mathbb{R}$. Note that Hamiltonian is unbounded and its spectrum $\text{Sp}(\widehat{H}_{\text{quant}}) = \mathbb{R}$. Moreover, the radial part

$$R_\lambda(r) = \frac{1}{\sqrt{2\pi}}r^{-(i\lambda+1)}\,,$$

does not belong to $L^2(\mathbb{R}_+, rdr)$ and should be treated as a distribution.[8]

As it was shown in [26, §5], distributions $\Psi_{l\lambda}$ provide spectral decomposition of the Bateman Hamiltonian (generalized to the space of distributions). Due to the invariance of the Hamiltonian with respect to the time reversal operation, the time-reversed distributions lead to another spectral decomposition. It is necessary to introduce two classes of test functions $\mathcal{S}_\pm \subset L^2(\mathbb{R})$, which naturally lead to two types of evolution: 'forward in time' and 'backward in time', thus leading to irreversibility.[9]

In addition, the poles of a resolvent operator

$$\widehat{\text{R}}(\widehat{H}_{\text{quant}}, z) = (\widehat{H}_{\text{quant}} - z)^{-1}$$

[8]Obviously the eigenfunction $\Psi_{l\lambda}(r,\varphi)$ are distributions as well and their action on a test function $\phi(r,\varphi) \in \mathcal{S}$ is the following:

$$\Psi_{l\lambda}(\phi) = \langle\phi|\Psi_{l\lambda}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-il\varphi}r^{-i\lambda-1}\overline{\phi}(r,\varphi)dS = \int_0^\infty r^{-i\lambda}\overline{\phi}_l(r)dr\,,$$

where $dS = rdrd\varphi$ and

$$\phi_l(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{il\varphi}\phi(r,\varphi)\,d\varphi\,.$$

[9]The sets $\mathcal{S}_\pm$ are defined as follows: a function $f^\pm\colon \mathbb{R}\to\mathbb{C}$ is in an upper (+) (lower (-)) Hardy class $\mathcal{H}^2_\pm$, if $f^\pm$ can be analytically continued to $F^\pm$ defined on the upper (lower) complex half-plane $\mathbb{C}^\pm$, such that for $K > 0$ [28]

$$\sup_y \int_{\mathbb{R}} |F^\pm(x \pm iy)|^2 dx < K$$

is valid. Then $\mathcal{S}_\pm$ is such a subset of test functions $\mathcal{S}$ that

$$\mathcal{S}_\pm = \{\phi \in \mathcal{S} \,:\, \langle\phi|\Psi_{l\lambda}\rangle \in \mathcal{H}^2_\pm\}\,,$$

in other words $\phi \in \mathcal{S}_\pm$, if the function $\mathbb{C} \ni \lambda \mapsto \langle\phi|\Psi_{l\lambda}\rangle \in \mathbb{C}$ belongs to the Hardy class $\mathcal{H}^2_\pm$.

defined on $\mathcal{S}_\pm$ read

$$E^\pm_{nl} = \hbar l\omega \pm i\hbar\gamma(|l| + 2n + 1)$$

and coincide after a suitable identification with the discrete family of complex energy eigenvalues $E^\pm_{jm}$ given by (46),

$$j = \frac{l}{2}\,, \qquad m = \frac{1}{2}(|l| + 2n) = |j| + n\,.$$

Since the eigenstates corresponding to the poles of a resolvent are known in a scattering theory as *resonant states*, our considerations lead to the conclusion that all the phenomena: resonances, dissipation and irreversibility of evolution are connected with each other. Moreover, our results indicate that the natural formalism for the mathematical description of this system is that of two Gelfand triples provided by $\mathcal{S}_\pm$ and their dual counterpart $\mathcal{S}'_\pm$ [29]:

$$\mathcal{S}_\pm \;\subset\; L^2(\mathbb{R}^2) \;\subset\; \mathcal{S}'_\pm\,.$$

For further discussion of this subject, in particular, for the relationship of the DHO system with two-dimensional parabolic potential barrier see [30].

The DHO equation may also be represented by a time-dependent Hamiltonian system. Indeed, Hamilton equations for the Caldirola-Kanai Hamiltonian

$$H_{\mathrm{CK}}(x,p,t) = \frac{1}{2}m_0\omega^2x^2\,e^{2\gamma t} + \frac{1}{2m_0}p^2\,e^{-2\gamma t}\,. \tag{51}$$

reproduce the DHO equation as well. Note that the function (51) is a potential energy of an harmonic oscillator with a mas term depending on time, i.e. $m(t) = m_0e^{2\gamma t}$, where $m_0 = m(0)$. This suggests to consider a general time-dependent Hamilton function with a mass depending on time, i.e.

$$H(x,p,t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2x^2\,.$$

Then the appropriate Newton equation reads

$$\ddot{x} + \frac{\dot{m}(t)}{m(t)}\,\dot{x} + \omega^2x = 0\,, \tag{52}$$

and we call it a *generalized damped harmonic oscillator equation* (GDHO) [31]. The problem we considered in [31] was the following: determine the mass term $m(t)$ which meets the following conditions:

(1) for large t the equation (52) transforms to DHO equation, i.e.,

$$\kappa(t) := \frac{1}{2}\frac{\dot{m}(t)}{m(t)} \longrightarrow \gamma\,,$$

(2) solutions to (52) describe the oscillations with a constant ω and its value is the same as the frequency of damped harmonic oscillations

$$\omega = \sqrt{\omega_0^2 - \gamma^2}\,.$$

We analysed also a possible quantisation of such systems.

It turns out that the conditions (1)–(2) are fulfilled in two cases *only*:

$$\kappa(t) = \gamma\,\mathrm{tgh}(\gamma(t - t_0)) \quad \text{ or } \quad \kappa(t) = \gamma\,,$$

where t_0 is an integration constant. These leads to

$$m(t) = m_0 \cosh^2(\gamma(t-t_0)) \quad \text{or} \quad m(t) = m_0\, e^{2\gamma(t-t_0)}$$

where in both models $m(t_0) = m_0$ is assumed. It is obvious that $m(t) = m_0 e^{2\gamma(t-t_0)}$ leads to the Caldirola-Kanai (CK) model. Its quantum version was extensively studied (see for instance [25]). But already at the classical level there is a significant difference in dynamical features of both models: the matrix $\Lambda^{\rm CK}(t,t_0)$ describing the dynamics in CK model

$$\left(\begin{array}{c} x(t) \\ p(t) \end{array}\right) = \Lambda^{\rm CK}(t;t_0) \left(\begin{array}{c} x_0 \\ p_0 \end{array}\right)$$

fulfils the composition law

$$\Lambda^{\rm CK}(t_2;t_0) = \Lambda^{\rm CK}(t_2;t_1) \circ \Lambda^{\rm CK}(t_1;t_0)$$

whereas an analogue matrix $\Lambda^{\rm GDHO}(t;t_0)$ describing the dynamics of the second model with $m(t) = m_0 \cosh^2(\gamma(t-t_0))$, *does not* obey the law! As a consequence, the parameter t_0 arising in $m(t) = m_0 \cosh^2(\gamma(t-t_0))$ is considered as responsible for memory effects (in a sense of non-Markovian dynamics). These results (discussed in more details in [31]) can be generalised to the quantum counterparts of systems under discussion. The memory effects arise then as a result of a violation of the composition law for quantum propagators

$$K(x,t;x_0,t_0) = \langle x|{\rm T}\exp\Big(-\frac{i}{\hbar}\int\limits_{t_0}^{t} H(t')dt'\Big)|x_0\rangle\,,$$

provided by a choice of Hamiltonian $H(t)$. A further generalisation of these observations in [32] enables to show that an (integral) evolution equation with memory

$$\frac{d}{dt}\Lambda(t,t_0) = \int\limits_{t_0}^{t} \mathcal{K}(t-\tau)\Lambda(\tau,t_0)d\tau$$

can be alternatively represented by a local-in-time differential equation

$$\frac{d}{dt}\Lambda(t,t_0) = \mathcal{L}(t-t_0)\Lambda(t,t_0)\,,$$

where, however, the information about the initial moment t_0 of evolution is built up in a generator $\mathcal{L}$. This property distinguishes the local-in-time equations with memory from those without memory.

Summing up the results of my research:

(1) using the method of Pontryagin and introducing some additional degrees of freedom we have defined a Hamiltonian for the damped harmonic oscillator model, which generates the correct equation of motion. We have shown that this Hamiltonian is equivalent to Bateman Hamiltonian,

(2) we have proposed a method of canonical quantisation of the resulting Hamiltonian system in polar variables, determining its eigenvalues and eigenvectors as well as the Feynman propagator. We have discussed the properties of its spectral decomposition resolvent,

(3) we have shown that the discrete eigenvalues of the Bateman Hamiltonian correspond to the poles of two resolvent operators defined on the spaces of test functions $\mathcal{S}_{\pm}$, and the eigenfunctions corresponding to these values are responsible for resonant states in the system. The spaces $\mathcal{S}_{\pm}$ and their duals $\mathcal{S}'_{\pm}$ are elements of two Gelfand triples that provide a natural environment for mathematical description of the DHO,

(4) we have observed that the appearance of resonant states leads to dissipative phenomena and irreversible dynamics,

(5) we have discussed more general equation with dissipation (52) containing the initial time t_0 and we have interpreted it as leading to memory effects in a local-in-time manner. We have shown that these observations are still correct at the quantum level and lead to loss of a composition law for quantum propagators.

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7 Leadership and participation in research projects

(1) Współwykonawca Grantu MENiS 3004/B/H03/2007/33,
Kwantowe splątanie: analiza i klasyfikacja, 2007–2010.

(2) Współwykonawca Grantu Narodowego Centrum Nauki,
projekt DEC-2011/03/B/ST2/00136,
Kwantowe korelacje: analiza, detekcja i dynamika

8 Oral communications and participation in conferences

8.1 Oral communications

(1) 40th Symposium on Mathematical Physics, 25-28 czerwca 2008, „Geometry and Quanta", referat: *Testing the new class of PPT states for entanglement*

(2) Satelite Conference of ICQBIC09, Suwa, Japan, 6–7.03.2009,
referat: *Introduction to quantisation of dissipative systems.*

(3) ICQBIC09 International Conference on Quantum BioInformatics, Tokyo University of Science, Noda, Japan, 11–14.03.2009,
referat: *Nonlocally damped harmonic oscillator and its quantisation.*

(4) International Conference QBIC10, Tokyo University of Science, Noda, Japan, 10–13.03.2010,
referat: *On estimations of entanglement measures.*

(5) 28th International Colloquium on Group-Theoretical Methods in Physics, Northumbria University, New Castle, Great Britain, 26–30.07.2010,
referat: *On estimations of entanglement measures*,

(6) International Conference on Quantum Bio-Informatics, Tokyo University of Science, Noda, Japan, 7–14.03.2011,
referat: *On local numerical ranges of operators*,

(7) XXX Workshop on Geometric Methods in Physics, Białowieża, Poland, 26.06–2.07.2011,
referat: *q-Discord for generalized entropies.*

(8) XXIX International Colloquium on Group-Theoretical Methods in Physics, Chern Institute of Mathematics, Tianjin, China, 20–26.08.2012,
poster: *Discord Derived from Tsallis Entropy for Werner and Isotropic States.*

8.2 Participation in conferences

(1) XXVI Symposium on Mathematical Physics, Toruń, 1993.

(2) Quantum Systems. New Trends and Method, Minsk, Belarus, 1994.

(3) VI Workshop Open Systems & Information Dynamics, Toruń, 1994.

(4) XXVII Symposium on Mathematical Physics, Toruń, 1994.

(5) Hamiltonian and Lagrangian Mechanics, Warszawa, 05.1995.

(6) Classical and Quantum Gauge Theory, Warszawa, 05.1995.

(7) Classical and Quantum Gravity, Warszawa, 05.1995.

(8) VI International Conference on Differential Geometry and Applications, Brno, 1995.

(9) XXVIII Symposium on Mathematical Physics, Toruń, 1995.

(10) VII Workshop Open Systems & Information Dynamics, Toruń, 1996.

(11) XXI International Colloquium on Group Theoretical Methods in Physics, Goslar, July 1996.

(12) XXIX Symposium on Mathematical Physics, Toruń, 1996.

(13) Fifth International Wigner Symposium, Wiedeń, 25-29.08.1997.

(14) VIII Workshop Open Systems & Information Dynamics, Toruń, 1998.

(15) XXX Symposium on Mathematical Physics, Toruń, 1998.

(16) XXXI Symposium on Mathematical Physics, 18-21 maja 1999, „Solitons and Nonlinear Phenomena"

(17) 32nd Symposium on Mathematical Physics, 6-10 czerwca 2000, „Symmetries in Nonlinear Systems"

(18) 33rd Symposium on Mathematical Physics, 5-9 czerwca 2001, „Nonholonomic Systems and Contact Structures"

(19) 34th Symposium on Mathematical Physics, 14-18 czerwca 2002, „Physical and Control-Theoretic Applications of Sub-Riemannian and Finsler Geometries"

(20) 35th Symposium on Mathematical Physics, 10-11 października 2003,

(21) 36th Symposium on Mathematical Physics, 9-12 czerwca 2004, „Open Systems and Quantum Information"

(22) 37th Symposium on Mathematical Physics, 17-18 czerwca 2005,

(23) 38th Symposium on Mathematical Physics, 4-7 czerwca 2006, „Quantum Entanglement and Geometry"

(24) 39th Symposium on Mathematical Physics, 11-12 czerwca 2007,

(25) 40th Symposium on Mathematical Physics, 25-28 czerwca 2008, „Geometry and Quanta",

(26) 41st Symposium on Mathematical Physics, Toruń, Poland, 5–6.06.2009.

(27) 12th Workshop: Non-Commutative Harmonic Analysis, Będlewo, Polska, 16–22.08.2009.

(28) Satelite Conference of ICQBIC09, referat: *Introduction to quantisation of dissipative systems*, Suwa, Japan, 6–7.03.2009.

(29) ICQBIC09 International Conference on Quantum BioInformatics, Tokyo University of Science, Noda, Japan, 11–14.03.2009.

(30) Symmetry and Structural Properties of Condensed Matter, Myczkowce, Poland, 2–9.09.2009.

(31) International Conference QBIC10, Tokyo University of Science, Noda, Japan, 10–13.03.2010.

(32) 42nd Symposium on Mathematical Physics, Toruń, Poland, 19–22.06.2010.

(33) 28th International Colloquium on Group-Theoretical Methods in Physics, Northumbria University, New Castle, Great Britain, 26–30.07.2010.

(34) International Conference on Quantum Bio-Informatics, Tokyo University of Science, Noda, Japan, 7–14.03.2011.

(35) 43rd Symposium on Mathematical Physics, Toruń, Poland, 20–22.06.2011.

(36) XXX Workshop on Geometric Methods in Physics, Białowieża, Poland, 26.06–2.07.2011.

(37) 44th Symposium on Mathematical Physics, Toruń, Poland, 20–24.06.2012.

(38) XXIX International Colloquium on Group-Theoretical Methods in Physics, Chern Institute of Mathematics, Tianjin, China, 20–26.08.2012.