Prawdziwe kwantowe oblicze "wykładniczego" prawa rozpadu: Cząstki nietrwałe w stanie spoczynku i w ruchu.

The true face of of the "exponential" decay: Unstable systems in rest and in motion

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Contens

Introduction

The radioactive decay law formulated by Rutherford and Sody in the nineteenth century [1, 2, 3] allows to determine the number $N(t)$ of atoms of the radioactive element at the instant t knowing the initial number $N_0 = N(0)$ of them at initial instant of time $t_0 = 0$ and has the exponential form:

$$N(t) = N_0 \, \exp{[-\lambda t]},$$

where $\lambda > 0$ is a constant. Since then, the belief that the decay law has the exponential form has become common.

This conviction was upheld by Wesisskopf–Wigner theory of spontaneous emission [4]: They found that to a good approximation the quantum mechanical non–decay probability of the exited levels is a decreasing function of time having exponential form.

Further studies of the quantum decay process showed that basic principles of the quantum theory does not allow it to be described by an exponential decay law at very late times [5, 6] and at initial stage of the decay process (see [6] and references therein). Theoretical analysis shows that at late times the survival probability (i. e. the decay law) should tends to zero as $t \to \infty$ much more slowly than any exponential function of time and that as function of time it has the inverse power–like form at this regime of time [5, 6]. All these results caused that there are rather widespread belief that a universal feature of the quantum decay process is the presence of three time regimes of the decay process: the early time (initial), exponential (or "canonical"), and late time having inverse–power law form [7]. This belief is reinforced by a numerous presentations in the literature of decay curves obtained for quantum models of unstable systems. In this context, each experimental evidence of oscillating decay curve at times of the order of life times is considered as an anomaly caused by a new quantum effects or new interactions: The so–called GSI–anomaly [8, 9] is an example.

Introduction

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Physics Letters B

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Observation of non-exponential orbital electron capture decays of hydrogen-like ^{140}Pr and ^{142}Pm ions

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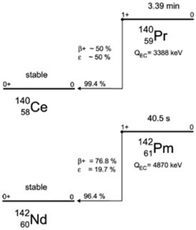

Fig. 1. Decay schemes of neutral ^{140}Pr (upper panel) and ^{142}Pm (lower panel) atoms [14].

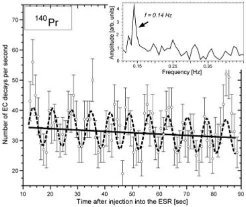

Fig. 3. Number of EC-decays of H-like ^{140}Pr ions per second as a function of the time after the injection into the ring. The solid and dashed lines represent the fits according to Eq. (1) (without modulation) and Eq. (2) (with modulation), respectively. The inset shows the Fast Fourier Transform of these data. A clear frequency signal is observed at 0.14 Hz (laboratory frame).

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Physics Letters B

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High-resolution measurement of the time-modulated orbital electron capture and of the β^+ decay of hydrogen-like $^{142}Pm^{60+}$ ions

Two-Body-Weak-Decays Collaboration

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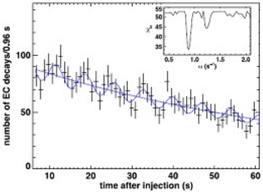

Fig. 2. Number of EC decays per 0.96 s of H-like $^{142}Pm^{60+}$ ions, recorded by the 245 MHz resonator, vs. the time after injection of the ions into the storage ring ESR. Displayed are also the exponential fit according to Eq. (1) and the modulation fit according to Eq. (2). The inset shows the χ^2 values vs. the angular frequency ω, for a fixed total decay constant λ and a variation of amplitude a and phase ϕ.

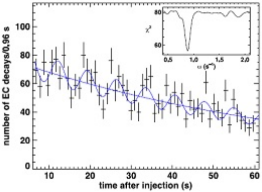

Fig. 3. Number of EC decays per 0.96 s of H-like $^{142}Pm^{60+}$ ions, recorded by the capacitive pick-up, vs. the time after injection of the ions into the storage ring ESR. Displayed are also the exponential fit according to Eq. (1) and the modulation fit according to Eq. (2). The inset shows the χ^2 values vs. the angular frequency ω, for a fixed total decay constant λ and a variation of amplitude a and phase ϕ.

The question arises, if indeed in the case of one component quantum unstable systems such oscillations of the decay process at the "exponential" regime are an anomaly, or perhaps universal feature of quantum decay processes.

The mentioned GSI anomaly is the cause of that the another question arises: Whether and how such a possible oscillations depend on the motion of the unstable quantum system. To find an answer to this question we need to know how to describe the decay process of unstable quantum systems in motion.

From the standard, text book considerations one finds that if the decay law of the unstable particle in rest has the exponential form

$$\mathcal{P}_0(t) = e^{-\frac{\Gamma_0\, t}{\hbar}},$$

then the decay law of the moving particle looks as follows

$$\mathcal{P}_p(t) \,=\, e^{-\,\frac{\Gamma_0\, t}{\hbar\,\gamma}}, \tag{1}$$

where t denotes time, Γ_0 is the decay rate (time t and Γ_0 are measured in the rest reference frame of the particle) and γ is the relativistic Lorentz factor, $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta = v/c$, $v = |\vec{v}|$ is the velocity of the particle, $\vec{v} = c\vec{p}/\sqrt{\vec{p}^2 + m_0^2\, c^2}$ and m_0 – is the rest mass. The equality (1) is the classical physics relation. It is almost common belief that this equality is valid also for any t in the case of quantum decay processes and does not depend on the model of the unstable particles considered.

The problem seems to be extremely important because from some theoretical studies it follows that in the case of quantum decay processes this relation is valid to a sufficient accuracy only for not more than a few lifetimes $\tau_0 = \hbar/\Gamma_0$ [10, 11, 12, 13]. On the other hand all known tests of the relation

$$\mathcal{P}_p(t) \,=\, e^{-\,\frac{\Gamma_0\,t}{\hbar\,\gamma}},$$

were performed for times of the order of τ_0 (see, eg. [14, 15]) and for times longer than a few lifetimes this relation was not tested till now.

So, in the following it will be shown that that in the case of unstable systems in rest and decaying in vacuum there is no time interval in which the survival probability (decay law) could be a decreasing function of time of the purely exponential form. We also show that even in the case of a single component unstable system the decay curve has an oscillatory form with a smaller or a large amplitude of oscillations depending on the model considered. In the following it will also be shown that the relativistic treatment of the problem within the Stefanovich–Shirokov theory [10, 11] yields decay curves tending to zero as $t \to \infty$ much slower than one would expect using classical time dilation relation which confirms and generalizes some conclusions drawn in [13]. Our results shows that conclusions relating to the quantum decay processes of moving particles based on the use of the classical physics time dilation relation need not be universally valid.

The main information about properties of quantum unstable systems is contained in their decay law, that is in their survival probability. Let the reference frame $\mathcal{O}_0$ be the common inertial rest frame for the observer and for the unstable system. Then if one knows that the system in the rest frame is in the initial unstable state $|\phi\rangle \in \mathcal{H}$, ($\mathcal{H}$ is the Hilbert space of states of the considered system), which was prepared at the initial instant $t_0 = 0$, one can calculate its survival probability (the decay law), $\mathcal{P}_0(t)$, of the unstable state $|\phi\rangle$ **decaying in vacuum**, which equals

$$\mathcal{P}_0(t) = |a_0(t)|^2, \tag{2}$$

where $a_0(t)$ is the probability amplitude of finding the system at the time t in the rest frame $\mathcal{O}_0$ in the initial unstable state $|\phi\rangle$,

$$a_0(t) = \langle \phi | \phi(t) \rangle. \tag{3}$$

and $|\phi(t)\rangle$ is the solution of the Schrödinger equation for the initial condition $|\phi(0)\rangle = |\phi\rangle$, which has the following form within the system units $\hbar = c = 1$ used in the next parts of this talk:

$$i \frac{\partial}{\partial t} |\phi(t)\rangle = H |\phi(t)\rangle. \tag{4}$$

Here $|\phi\rangle, |\phi(t)\rangle \in \mathcal{H}$, and H denotes the total self–adjoint Hamiltonian for the system considered. Note that if $|\phi\rangle$ represents an unstable state then it cannot be an eigenvector for H: In such a case the eigenvalue equation $H|\phi\rangle = \epsilon_\phi |\phi\rangle$ has no solutions for $|\phi\rangle$ under considerations.

There is $|\phi(t)\rangle = U(t)|\phi\rangle$, where $U(t) = \exp\,[-itH]$ is unitary evolution operator and $U(0) = \mathbb{I}$ is the unit operator. Operators H and $U(t)$ have common eigenfunctions.

The rest reference frame $\mathcal{O}_0$ is defined using common solution of the eigenvalue problem for H and the momentum operator $\mathbf{P}$:

$$\mathbf{P}|\mu; p\rangle = \vec{p}|\mu; p\rangle, \tag{5}$$

and

$$H|\mu; p\rangle = E'(\mu, p)\,|\mu; p\rangle, \tag{6}$$

where $\mu \equiv E'(\mu, 0) \in \sigma_c(H)$ and $\sigma_c(H)$ is the continuous part of the spectrum of the Hamiltonian H. Operators H and $\mathbf{P}$ act in the state space $\mathcal{H}$. There is (see [10, 11, 16, 17]),

$$E'(\mu, p) \equiv \sqrt{\mu^2 + (\vec{p})^2}. \tag{7}$$

In the rest reference frame of the quantum unstable system $\mathcal{O}_0$, when $\vec{p} = 0$, we have $|\mu; 0\rangle = |\mu; p = 0\rangle$,

$$\mathbf{P}|\mu; 0\rangle = 0, \tag{8}$$

and

$$H|\mu; 0\rangle = \mu\,|\mu; 0\rangle, \quad \mu \in \sigma_c(H), \tag{9}$$

Eigenvectors $|\mu; 0\rangle$ are normalized as usual:

$$\langle 0; \mu|\mu'; 0\rangle = \delta(\mu - \mu'). \tag{10}$$

Now we can model the unstable system in the rest system $\mathcal{O}_0$ as the following wave–packet $|\phi_0\rangle \equiv |\phi_{\vec{p}=0}\rangle \stackrel{\text{def}}{=} |\phi\rangle$,

$$|\phi_0\rangle \equiv |\phi\rangle = \int_{\mu_0}^{\infty} c(\mu)\, |\mu; 0\rangle \, d\mu, \tag{11}$$

where expansion coefficients $c(\mu)$ are functions of the mass parameter μ, that is of the rest mass μ. (Here μ_0 is the lower bound of the spectrum $\sigma_c(H)$ of H). We require the state $|\phi_0\rangle$ to be normalized: So it has to be

$$\int_{\mu_0}^{\infty} |c(\mu)|^2 \, d\mu = 1. \tag{12}$$

The expansion (11) and relation (9) allow one to find the amplitude $a_0(t)$ and to write [6, 18]

$$a_0(t) \equiv \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\,\mu\,t}\, d\mu, \tag{13}$$

where $\omega(\mu) \equiv |c(\mu)|^2 > 0$.

So the amplitude $a_0(t)$, and thus the decay law $\mathcal{P}_0(t)$ of the unstable state $|\phi\rangle$, are completely determined by the density of the mass (energy) distribution $\omega(\mu)$ for the system in this state [18] (see also: [5, 6, 19, 20, 21, 22, 23]. From (13) and from the Riemann–Lebesque lemma it follows that $|a(t)| \to 0$ as $t \to \infty$. It is because from the normalization condition (12) it follows that $\omega(\mu)$ is an absolutely integrable function. (Note that this approach is also applicable in Quantum Field Theory models [24, 25]).

Khalfin in his paper [5] published in 1957 assuming that the spectrum of H must be bounded from below, $\mu_0 > -\infty$), and using the Paley–Wiener Theorem [26] proved that in the case of unstable states there must be

$$|a_0(t)| \;\geq\; A \exp\left[-b\, t^q\right],$$

for $|t| \to \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$. Therefore the decay law $\mathcal{P}_0(t)$ of unstable states decaying in the vacuum, (2), can not be described by an exponential function of time t if time t is suitably long, $t \to \infty$, and that for these lengths of time $\mathcal{P}_0(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of t. This this effect was confirmed in experiment described in the Rothe paper [27]:

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Violation of the Exponential-Decay Law at Long Times

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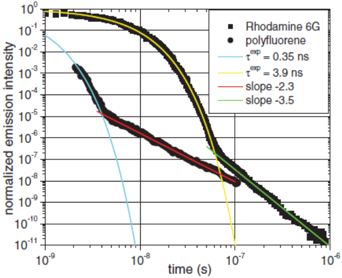


FIG. 2 (color). Corresponding double logarithmic fluorescence decays of the emissions shown in Fig. 1. Exponential and power law regions are indicated by solid lines and the emission intensity at time zero has been normalized.

Note that the use of the Schrödinger equation (4) allows one to find that within the problem considered.

$$i\frac{\partial}{\partial t}\langle\phi|\phi(t)\rangle = \langle\phi|H|\phi(t)\rangle. \tag{14}$$

This relation leads to the conclusion that the amplitude $a_0(t)$ satisfies the following equation

$$i\frac{\partial a_0(t)}{\partial t} = h(t)\, a_0(t), \tag{15}$$

where

$$h(t) = \frac{\langle\phi|H|\phi(t)\rangle}{a_0(t)}, \tag{16}$$

and $h(t)$ is the effective Hamiltonian governing the time evolution in the subspace of unstable states $\mathcal{H}_{||} = P\mathcal{H}$, where $P = |\phi\rangle\langle\phi|$ (see [28] and also [29, 30] and references therein).

The subspace $\mathcal{H} \ominus \mathcal{H}_{\|} = \mathcal{H}_{\perp} \equiv Q\mathcal{H}$ is the subspace of decay products. Here $Q = \mathbb{I} - P$. There is the following equivalent formula for $h(t)$ [28, 29, 30]:

$$h(t) \equiv \frac{i}{a_0(t)} \frac{\partial a_0(t)}{\partial t}. \tag{17}$$

If $\langle\phi|H|\phi\rangle$ exists then using unitary evolution operator $U(t)$ and projection operators P and Q the relation (16) can be rewritten as follows

$$h(t) = \langle\phi|H|\phi\rangle \; + \; \frac{\langle\phi|HQ\,U(t)|\phi\rangle}{a_0(t)}. \tag{18}$$

One meets the effective Hamiltonian $h(t)$ when one starts with the Schrödinger equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$ [28, 19].

In general $h(t)$ is a complex function of time and in the case of $\mathcal{H}_{\|}$ of dimension two or more the effective Hamiltonian governing the time evolution in such a subspace it is a non–hermitian matrix $H_{\|}$ or non-hermitian operator. There is

$$h(t) = \mu_{\phi}(t) - \frac{i}{2}\gamma_{\phi}(t), \tag{19}$$

and

$$\mu_{\phi}(t) = \Re\,[h(t)], \qquad \gamma_{\phi}(t) = -\,2\,\Im\,[h(t)], \tag{20}$$

are the instantaneous mass (energy) $\mu_{\phi}(t)$ and the instantaneous decay rate, $\gamma_{\phi}(t)$ [28, 29, 30]. Here $\Re\,(z)$ and $\Im\,(z)$ denote the real and imaginary parts of z respectively. The relations (15), (17) and (20) are convenient when the density $\omega(\mu)$ is given and one wants to find the instantaneous mass $\mu_{\phi}(t)$ and decay rate $\gamma_{\phi}(t)$: Inserting $\omega(\mu)$ into (13) one obtains the amplitude $a_0(t)$ and then using (17) one finds the $h(t)$ and thus $\mu_{\phi}(t)$ and $\gamma_{\phi}(t)$.
From (18) it follows that

$$\mu_{\phi}(0) = \langle \phi | H | \phi \rangle, \quad \text{and} \quad \gamma_{\phi}(0) = 0.$$

Note that the state vector $|\phi\rangle$ of the form (11) corresponding to a quantum unstable system **can not be an eigenvector of the Hamiltonian** H, otherwise it would be that

$$\mathcal{P}_0(t) = |\langle\phi|\phi(t)\rangle|^2 = |\langle\phi|\exp[-itH]\phi\rangle|^2 \equiv 1$$

for all times t.

The fact that the vector $|\phi\rangle$ describing the unstable quantum system is not the eigenvector for H means that the mass (energy) of this object is not defined. Simply the mass can not take the exact constant value in this state $|\phi\rangle$. In such a case quantum systems are characterized by the mass (energy) distribution density $\omega(\mu)$ and the average mass

$$< m >= \int_{\mu_0}^{\infty} \mu\, \omega(\mu)\, d\mu$$

or by the instantaneous mass (energy) $\mu_\phi(t)$ but not by the exact value of the mass.

The simplest way to compare the decay law $\mathcal{P}_0(t)$ with the exponential (canonical) decay law $\mathcal{P}_c(t)$, where $\mathcal{P}_c(t) = |a_c(t)|^2$ and

$$a_c(t) = \exp\,[-i\frac{t}{\hbar}(m_\phi - \frac{i}{2}\Gamma_\phi], \tag{21}$$

(where Γ_ϕ is the decay rate) is to analyze properties of the following function:

$$\zeta(t) \stackrel{\text{def}}{=} \frac{a_0(t)}{a_c(t)}. \tag{22}$$

There is

$$|\zeta(t)|^2 = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)}. \tag{23}$$

Analysis of properties of this function allows one to visualize all the more subtle differences between $\mathcal{P}_0(t)$ and $\mathcal{P}_c(t)$.

We have

$$\frac{\partial \zeta(t)}{\partial t} \equiv \frac{i}{\hbar}(m_\phi - \frac{i}{2}\Gamma_\phi)\,\zeta(t) + e^{+i\frac{t}{\hbar}(m_\phi - \frac{i}{2}\Gamma_\phi)}\,\frac{\partial a(t)}{\partial t}$$
$$= \frac{i}{\hbar}(m_\phi - \frac{i}{2}\Gamma_\phi)\,\zeta(t) - \frac{i}{\hbar}\,h(t)\,\zeta(t), \qquad (24)$$

where $h(t)$ is the effective Hamiltonian defined by relations (16) — (18)

Let us use now the relation (18) and assume that $\langle\phi|H|\phi\rangle$ exists and there exists instants $0 < t_1 < t_2 < \infty$ of time t such that for any $t \in (t_1, t_2)$ there is

$$\zeta(t) = \zeta(t_1) = \zeta(t_2) = const \stackrel{\text{def}}{=} c_\phi \neq 0. \qquad (25)$$

In such a case there should be $\frac{\partial \zeta(t)}{\partial t} = 0$ for all $t \in (t_1, t_2)$. Taking into account that by definition $\zeta(t) \neq 0$ from (24) we conclude that it is possible only and only if

$$h(t) - (m_\phi - \frac{i}{2}\Gamma_\phi) = 0, \quad \text{for} \;\; t_1 \leq t \leq t_2, \tag{26}$$

that is, if and only if

$$h(t_1) = h(t) = h(t_2) = const \stackrel{\text{def}}{=} c_h \neq 0 \quad \text{for} \;\; t_1 \leq t \leq t_2. \tag{27}$$

Using (18) and the property $|\phi(t)\rangle = U(t)\,|\phi\rangle$ one concludes that the equality $h(t_1) = h(t) = c_h$ can take place if

$$\frac{\langle\phi|HQ\,U(t_1)|\phi\rangle}{a_0(t_1)} = \frac{\langle\phi|HQ\,U(t)|\phi\rangle}{a_0(t)}. \tag{28}$$

Taking into account group properties of the one–parameter family of unitary operators $U(t)$ we can use in (28) $U(t_1)\,U(t-t_1) \equiv U(t)$ instead of $U(t)$. Next keeping in mind that $a_0(t) \neq 0,\; a_0(t_1) \neq 0$ and taking into account that $\lambda(t, t_1) \stackrel{\text{def}}{=} \frac{a_0(t)}{a_0(t_1)}$ is a complex function one can replace the relation (28) by the following one

$$\langle\phi|HQ\,U(t_1)\left[\lambda(t,t_1)|\phi\rangle \;-\; U(t-t_1)|\phi\rangle\right] = 0. \qquad (29)$$

This condition can be satisfied in two cases: The first one is

$$U(t-t_1)|\phi\rangle \;-\; \lambda(t,t_1)|\phi\rangle \;=\; 0, \qquad (30)$$

and the second one occurs when

$$[\,\lambda(t,t_1)|\phi\rangle \;-\; U(t-t_1)|\phi\rangle] \neq 0$$

together with

$$(\langle\phi|H)^{+} = H|\phi\rangle \;\;\perp\;\; Q\,U(t_1)\,[\lambda(t,t_1)|\phi\rangle \;-\; U(t-t_1)|\phi\rangle.$$

The first case means that $h(t_1) = h(t) = c_h = const$ which by (27) means that $\frac{\partial\zeta(t)}{\partial t} = 0$ if and only if the vector $|\phi\rangle$ representing an unstable state of the system is an eigenvector for the unitary evolution operator $U(t)$. As we noted earlier this operator $U(t)$ and the total Hamiltonian H of the system have common eigenvectors. This means that $h(t_1) = h(t) = c_h = const$ and thus $\frac{\partial\zeta(t)}{\partial t} = 0$ for $t \in (t_1, t_2)$ if and only if the unstable state $|\phi\rangle$ of the system is an eigenvector for H, which is in contradiction with the property that the vector $|\phi\rangle$ representing the unstable state cannot be the eigenvector for the total Hamiltonian H.

The second case: From the definition of the projectors P and Q it follows that this case can be realized only if the vector $H|\phi\rangle$ is proportional to the vector $|\phi\rangle$: $H|\phi\rangle = \alpha_\phi|\phi\rangle$, that is similarly to the first case $\frac{\partial\zeta(t)}{\partial t} = 0$ if and only if the vector $|\phi\rangle$ representing the unstable state of the system considered is an eigenvector for the total Hamiltonian H, which is again in clear contradiction with the condition that the vector $|\phi\rangle$ representing the unstable state cannot be the eigenvector for the total Hamiltonian H.

Taking into account implications of the above to possible realizations of the relation (29) we conclude the supposition that such time interval $[t_1, t_2]$ can exist that $h(t_1) = h(t) = c_h = const$ for $t \in (t_1, t_2)$ and thus $\zeta(t) = const = \zeta(t_1) = \zeta(t_2)$ for $t \in (t_1, t_2)$ is false. So taking into account the definition of $\zeta(t)$ the following conclusion follows: Within the approach considered in this paper for any time interval $[t_1, t_2]$ the decay law can not be described by the exponential function of time. This conclusion is the general one. It does not depend on models of quantum unstable states.

The another important conclusion is that at any time interval $[t_1, t_2]$ the effective Hamiltonian $h(t)$ can not be constant. This means that at any time interval $[t_1, t_2]$ the instantaneous mass $\mu_\phi(t) = \Re\,[h(t)]$ in the rest system $\mathcal{O}_0$ and decay rate $\gamma_\phi(t) = -2\Im\,[h(t)]$ can not be constant in time:

$$\mu_\phi(t) \neq const., \qquad \gamma_\phi(t) \neq const. \tag{31}$$

In other words, as it follows from the above analysis the case $\mu_\phi(t) = const$ and $\gamma_\phi(t) = const$ can be realized only if the state $|\phi\rangle$ is an eigenvector for the total Hamiltonian H, that is if an only if there is no any decay od the state $|\phi\rangle$.

This part of the talk was based, among others, on on the following papers [31, 32]:

- K. Urbanowski, True quantum face of the "exponential" decay law, *Eur. Phys. J. D*, (2017) **71**: 118.
- K. Urbanowski, On the Velocity of Moving Relativistic Unstable Quantum Systems, *Advances in High Energy Physics* **2015**, Article ID 461987.

Numerical studies: The Breit–Wigner model

In general the spectral density $\omega(\mu)$ has properties similar to the scattering amplitude, i.e., it can be decomposed into a threshold factor, a pole-function $P(m)$ with a simple pole (often modeled by a Breit-Wigner) and a smooth form factor $F(\mu)$. So, we can write

$$\omega(\mu) = \Theta(\mu - \mu_0)\,(\mu - \mu_0)^{\alpha_l}\, P(\mu)\, F(\mu), \tag{32}$$

where α_l depends on the angular momentum l through $\alpha_l = \alpha + l$, [6] (see equation (6.1) in [6]), $0 \leq \alpha < 1$) and $\Theta(\mu)$ is a step function: $\Theta(\mu) = 0 \;\text{ for }\; \mu \leq 0$ and $\Theta(\mu) = 1 \;\text{ for }\; \mu > 0$. The simplest choice is to take $\alpha = 0, l = 0, F(\mu) = 1$ and to assume that $P(\mu)$ has a Breit–Wigner form. It turns out that the decay curves obtained in this simplest case are very similar in form to the curves calculated for more general $\omega(\mu)$ defined by (32) (see [20] and analysis in [6]). So to find the most typical properties of the decay curve it is sufficient to make the relevant calculations for $\omega(\mu)$ modeled by the the Breit–Wigner distribution of the energy density $\omega(\mu) \equiv \omega_{BW}(\mu)$:

$$\omega_{BW}(\mu) = \frac{N}{2\pi}\,\Theta(\mu-\mu_0)\,\frac{\Gamma_0}{(\mu-m_0)^2+(\frac{\Gamma_0}{2})^2}, \tag{33}$$

where N is a normalization constant and $\Theta(\mu)$ is a step function.

Inserting (33) into (13) one can find analytical expression for $a_0(t)$ (see, eg. [29, 30, 33]):

$$\begin{aligned} a_0(t) &= N\,e^{-\frac{i}{\hbar}(m_0 - i\frac{\Gamma_0}{2})t}\times \\ &\times\Big\{1-\frac{i}{2\pi}\Big[e^{\frac{\Gamma_0 t}{\hbar}}\,E_1\Big(-\frac{i}{\hbar}(m_R+\frac{i}{2}\Gamma_0)t\Big) \\ &\quad +(-1)E_1\Big(-\frac{i}{\hbar}(m_R-\frac{i}{2}\Gamma_0)t\Big)\Big]\Big\}, \end{aligned} \tag{34}$$

where $E_1(x)$ denotes the integral–exponential function defined according to [34, 35] and $m_R = m_0 - \mu_0$.

From the last formula one finds that there is for $t \to \infty$:

$$a_0(t)\big|_{t\to\infty} \simeq \frac{N}{2\pi}\, e^{-\frac{i}{\hbar}\,\mu_0 t}\, \Big\{(-i)\,\frac{\Gamma_0}{|\,h_0 - \mu_0\,|^2}\,\frac{\hbar}{t} - 2\,\frac{(m_0 - \mu_0)\,\Gamma_0}{|\,h_0 - \mu_0\,|^4}\,\Big(\frac{\hbar}{t}\Big)^2 + \ldots\Big\} \qquad (35)$$

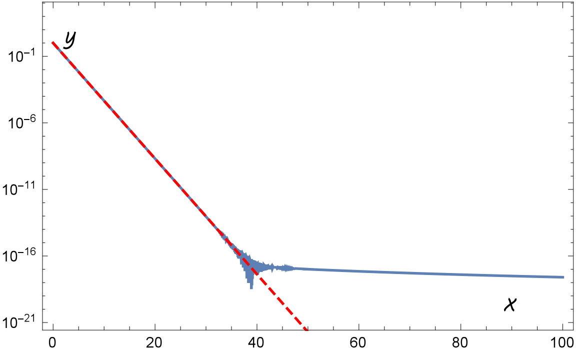
where $h_0 = m_0 - \frac{i}{2}\,\Gamma_0$, and

$$\mu_\phi(t)\big|_{t\to\infty} \simeq \mu_0 - 2\,\frac{m_0 - \mu_0}{|\,h_0 - \mu_0\,|^2}\,\Big(\frac{\hbar}{t}\Big)^2 + \ldots, \qquad (36)$$

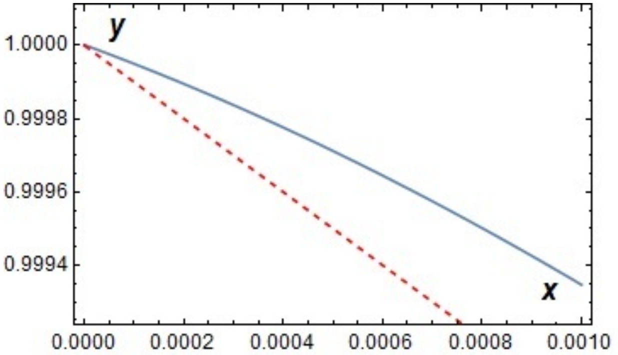
where $\mu_\phi(t) = \Re\,[h(t)]$ and $h(t)$ is defined by (16), (17).

Now analytical results presented above will be illustrated graphically. The typical form of the survival probability $\mathcal{P}_0(t)$ is presented in Fig (1). Numerical calculations were made for the distribution of the mass (energy) density $\omega(\mu)$ having the Breit–Wigner form (33).

The form of the decay curves depend on the ratio $s_R = \frac{m_R}{\Gamma_0}$: The smaller s_R, the shorter the time when the late time deviations form the exponential form of $\mathcal{P}_0(t)$ begin to dominate.



Rysunek: (1) Decay curves obtained for $\omega_{BW}(E)$ given by Eq. (33). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities on a logarithmic scale (The solid line — the decay curve $\mathcal{P}_0(t) = |a_0(t)|^2$; The dotted line — the canonical decay curve $\mathcal{P}_c(t) = |a_c(t)|^2$. The case $s_R = \frac{m_R}{\Gamma_0} = 1000$.



Rysunek: (2) Decay curves obtained for $\omega_{BW}(E)$ given by Eq. (33). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities on a logarithmic scale (The solid line — the decay curve $\mathcal{P}_0(t) = |a_0(t)|^2$; The dotted line — the canonical decay curve $\mathcal{P}_c(t) = |a_c(t)|^2$. The case $s_R = \frac{m_R}{\Gamma_0} = 1000$.

Within the considered model the standard canonical form of the survival amplitude $a_c(t)$, is given by the following relation,

$$a_c(t) = \exp\left[-i\frac{t}{\hbar}\left(m_0 - \frac{i}{2}\Gamma_0\right)\right]. \tag{37}$$

Γ_0 is the decay rate and $\frac{\hbar}{\Gamma_0} \equiv \frac{1}{\Gamma_0} = \tau_0$ is the lifetime within the assumed system of units $\hbar = c = 1$ (time t and Γ_0 are measured in the rest reference frame of the particle),

$$\mathcal{P}_c(t) = |a_c(t)|^2 \equiv e^{-\frac{\Gamma_0}{\hbar}\,t}, \tag{38}$$

is the canonical form of the survival amplitude.

The case $\omega(\mu) = \omega_{BW}(\mu)$ is the typical case considered in numerous papers and used therein to model decay processes. Therefore it is very important to analyze real form of the decay curves obtained using $\omega(\mu) = \omega_{BW}(\mu)$ and this is why we consider this case here.

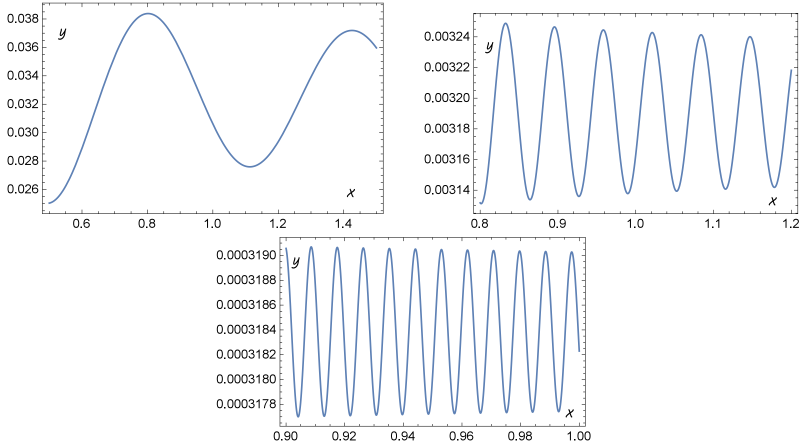
As already noted it is convenient to consider the function

$$\zeta(t) \stackrel{\text{def}}{=} \frac{a_0(t)}{a_c(t)}.$$

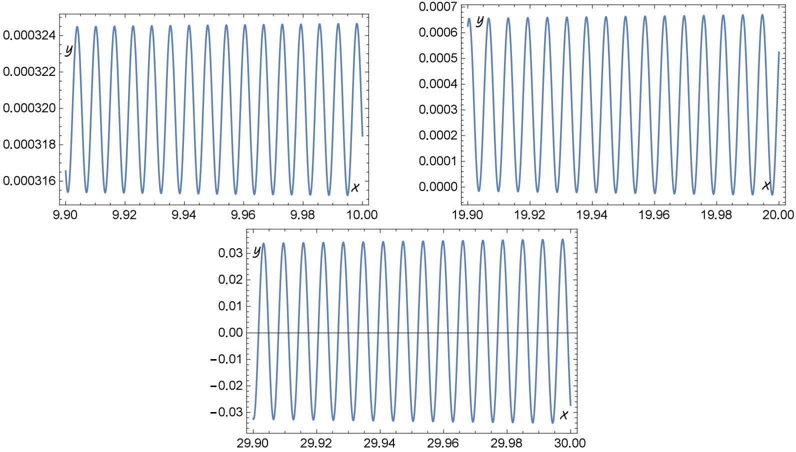
This is because

$$|\zeta(t)|^2 = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)},$$

Analysis of properties of this function allows one to visualize all the more subtle differences between $\mathcal{P}_0(t)$ and $\mathcal{P}_c(t)$. This function was used to find numerically $|\zeta(t)|^2$ for $\omega(m) = \omega_{BW}(m)$. Results of numerical calculations are presented in Figs (3) and (4): It turns out that in the case considered the form of $|\zeta(t)|^2$ also depend on the ratio $s_R \stackrel{\text{def}}{=} \frac{m_R}{\Gamma_0} \equiv \frac{m_0 - \mu_0}{\Gamma_0}$.



Rysunek: (3) A comparison of decay curves obtained for $\omega_{BW}(\mu)$ given by Eq. (33) with canonical decay curves. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — The function $f(t) = (|\zeta(t)|^2 - 1) = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)} - 1$, where $\zeta(t)$ is defined by the formula (22). The left panel: $s_R = 10$. The right panel: $s_R = 100$. The lower panel: $s_R = 1000$.

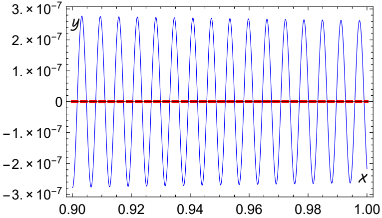


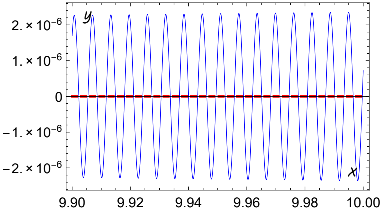
Rysunek: (4) A comparison of decay curves obtained for $\omega_{BW}(\mu)$ given by Eq. (33) with canonical decay curves. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — The function $f(t) = (|\zeta(t)|^2 - 1) = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)} - 1$, where $\zeta(t)$ is defined by the formula (22), $\mathcal{P}_0(t) = |a_0(t)|^2$, $\mathcal{P}_c(t) = |a_c(t)|^2$. The case $s_R = 1000$.

From the analysis performed in the previous Section it follows that the case $\mu_\phi(t) = const$ and $\gamma_\phi(t) = const$ can be realized only if the state $|\phi\rangle$ is an eigenvector for the total Hamiltonian H, that is if an only if there is no any decay od the state $|\phi\rangle$. Results of numerical calculations performed for $\omega(\mu) = \omega_{BW}(\mu)$ and presented in Fig (5) confirm such a conclusion, that is the conclusion denoted as (31). In these Figures the function

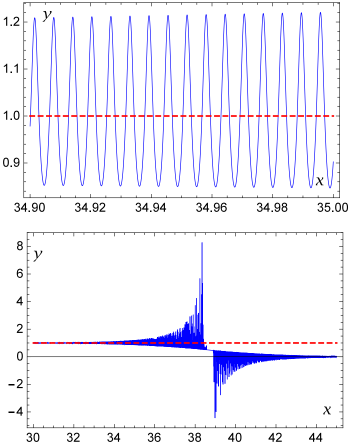
$$\kappa(t) = \frac{\mu_\phi(t) - \mu_0}{m_0 - \mu_0}, \tag{39}$$

is presented and calculations were performed for $s_R = \frac{m_R}{\Gamma_0} = \frac{m_0 - \mu_0}{\Gamma_0} = 1000$. The function $\kappa(t)$ illustrates a typical behavior of time-varying $\mu_\phi(t)$.





Rysunek: (5) The instantaneous mass $\mu_\phi(t)$ as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \kappa(t) - 1$, where $\kappa(t)$ is defined by (39); $x = t/\tau_\phi$: Time is measured in lifetimes. The horizontal dashed line represents the value of $\mu_\phi(t) = m_0$.



Rysunek: (6) The instantaneous mass $\mu_\phi(t)$ as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \kappa(t)$, where $\kappa(t)$ is defined by (39); $x = t/\tau_\phi$: Time is measured in lifetimes. The horizontal dashed line represents the value of $\mu_\phi(t) = m_0$. $s_R = \frac{m_0 - \mu_0}{\Gamma_0} = 1000$.

Summing up the oscillating decay curves of one component unstable system can not be considered as something extraordinary or as anomaly: It seems to be a universal feature of the decay process. The oscillatory modulation of decay curves takes place even in the quantum unstable system modeled by the Breit–Wigner distribution of the energy density. In general, the oscillatory modulation of the survival probability and thus the decay curves with model depending amplitude and oscillations period takes place even in the case of one component unstable systems. From results of the model calculations presented in Figs (3) and (4) it follows that at the initial stage of the "exponential" (or "canonical") decay regime the amplitude of these oscillations may be much less than the accuracy of detectors. Then with increasing time the amplitude of oscillations grows (see Fig. (4)), which increases the chances of observing them. This is a true quantum picture of the decay process at the so–called "exponential" regime of times.

The above part of the talk was based on on the following papers [31, 32, 36, 37]:

- K. Urbanowski, General properties of the evolution of unstable states at long times, *Eur. Phys. J. D*, **54**, 25, (2009).
- K. Urbanowski, True quantum face of the "exponential" decay law, *Eur. Phys. J. D*, (2017) **71**: 118.
- K. Urbanowski, The true face of quantum decay processes: Unstable systems at rest and in motion, *Acta Physica Polonica B*, **48** , 1847, (2017).
- K. Urbanowski, On the Velocity of Moving Relativistic Unstable Quantum Systems, *Advances in High Energy Physics* **2015**, Article ID 461987.

and others.

Moving unstable systems

Analyzing moving unstable systems one can follow the classical physics results and to assume that the unstable systems moves with the constant velocity $\vec{v}$, or guided by conservations laws to assume the momentum $\vec{p}$ of the moving unstable system is constant in time. The assumption $\vec{v} = const$ was used, eg. by Exner [12]. Exner obtained result that coincides with the classical result $\mathcal{P}_v(t) \simeq \mathcal{P}_0(t/\gamma)$ but detailed analysis shows that this results was obtained assuming that the velocity $\vec{v}$ is very small.
The second possibility to assume that $\vec{p} = const$ used by, e.g. Stefanovich [10] or Shirkov [11] leads to the results which does not depend on that whether the assumed momentum $\vec{p} = const$ is small or not.

Let us consider now the case of moving quantum system with definite momentum. We need the probability amplitude $a_p(t) = \langle \phi_p | \phi_p(t) \rangle$, which defines the survival probability

$$\mathcal{P}_p(t) = |a_p(t)|^2.$$

There is

$$|\phi_p(t)\rangle \stackrel{\text{def}}{=} \exp\left[-itH\right] |\phi_p\rangle$$

in $\hbar = c = 1$ units. So we need the vector $|\phi_p\rangle$ and eigenvalues $E'(\mu, p)$ solving Eq. (6). Vectors $|\phi\rangle, |\phi_p\rangle$ are elements of the same state space $\mathcal{H}$ connected with the coordinate rest system of the observer $\mathcal{O}$: We are looking for the decay law of the moving particle measured by the observer $\mathcal{O}$. If to assume for simplicity that $\mathbf{P} = (P_1, 0, 0)$ and that $\vec{v} = (v_1, 0, 0) \equiv (v, 0, 0)$ then there is $\vec{p} = (p, 0, 0)$ for the eigenvalues $\vec{p}$ of the momentum operator $\mathbf{P}$ and $|\vec{p}| = p$. Hence (see [10, 11, 16, 17]),

$$H|\mu; p\rangle = \sqrt{p^2 + \mu^2}\, |\mu; p\rangle \equiv \mu\, \gamma_\mu |\mu; p\rangle, \tag{40}$$

which replaces Eq. (6). Here $\gamma_\mu \equiv \frac{\sqrt{p^2+\mu^2}}{\mu}$.

In this idealized situation the moving quantum unstable particle ϕ with definite momentum, $\vec{p}$, can be modeled analogously as the quantum unstable system in the rest frame (when $\vec{p} = 0$) as the following wave–packet $|\phi_p\rangle$,

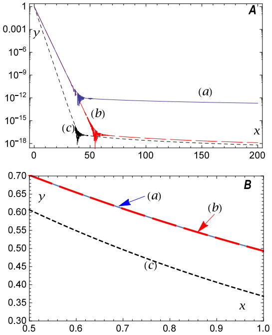
$$|\phi_p\rangle = \int_{\mu_0}^{\infty} c(\mu)\,|\mu; p\rangle\, d\mu, \tag{41}$$

where expansion coefficients $c(\mu)$ are functions of the mass parameter μ, that is of the rest mass μ, which is Lorentz invariant and therefore the scalar functions $c(\mu)$ of μ are also Lorentz invariant and are the same as in the rest reference frame $\mathcal{O}_0$.

Using (40) and the equation (41) we obtain the final relation for the amplitude $a_p(t)$ (see [10, 11, 17]),

$$a_p(t) \;\equiv\; \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\sqrt{p^2 + \mu^2}\;t}\, d\mu. \tag{42}$$

Results of numerical calculations are presented in Fig (7), where calculations were performed for $\omega(\mu) = \omega_{BW}(\mu)$ and $\mu_0 = 0$, $E_0/\Gamma_0 \equiv m_0/\Gamma_0 = 1000$ and $cp/\Gamma_0 \equiv p/\Gamma_0 = 1000$. Values of these parameters correspond to $\gamma = \sqrt{2}$, which is very close to γ from the experiment performed by the GSI team [8, 9] and this is why such values of them were chosen in our considerations. According to the literature for laboratory systems a typical value of the ratio m_0/Γ_0 is $m_0/\Gamma_0 \geq O(10^3 - 10^6)$ (see eg. [38]) therefore the choice $m_0/\Gamma_0 = 1000$ seems to be reasonable minimum.

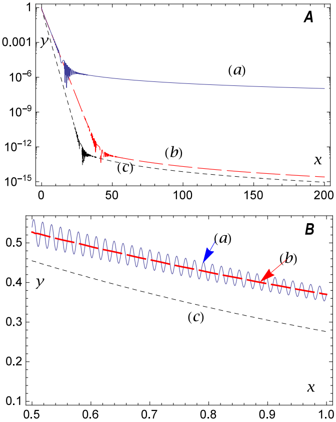


Rysunek: (7) Decay curves obtained for $\omega_{BW}(\mu)$ given by Eq. (33). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities (panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; panel B: (a) – $\mathcal{P}_p(t)$, (b) – $\mathcal{P}_0(t/\gamma)$, (c) – $\mathcal{P}_0(t)$).

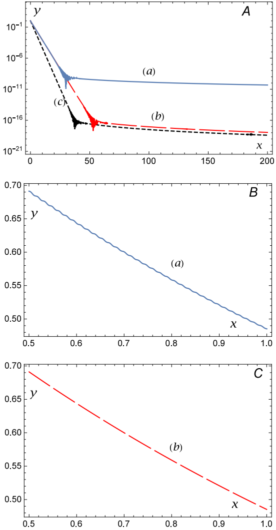
As it was mentioned earlier the formula (32) represents the general form of $\omega(\mu)$. Guided by this observation we follow [20] and assume that

$$\omega(\mu) = N\,\Theta(\mu - \mu_0)\,\sqrt{\mu - \mu_0}\,\frac{\sqrt{\Gamma_0}}{(\mu - m_0)^2 + (\Gamma_0/2)^2}\,e^{-\eta\,\frac{\mu}{m_0 - \mu_0}}, \quad (43)$$

with $\eta > 0$. Decay curves corresponding to this $\omega(\mu)$ were find numerically for the case of the particle decaying in the rest system (the survival probability $\mathcal{P}_0(t)$) as well as for the moving particle (the non–decay probability $\mathcal{P}_p(t)$). Results are presented in Figs (8) and (9). In order to compare them with the results obtained for $\omega_{BW}(\mu)$, calculations were performed for the same ratios as in that case: $m_0/\Gamma_0 = p/\Gamma_0 = 1000$, and $\mu_0 = 0$. The ratio $\eta\Gamma_0/(m_0 - \mu_0) \equiv \eta\Gamma_0/m_0$ was chosen to be $\eta\Gamma_0/m_0 = 0.01$ (Fig. (8)) and $\eta\Gamma_0/m_0 = 0.006$ (Fig. (9)).

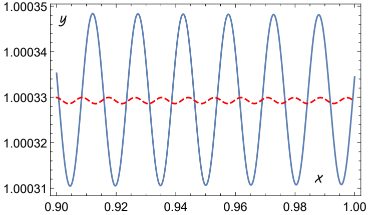


Rysunek: (8) Decay curves obtained for $\omega(\mu)$ given by Eq. (43). Axes: $x = t/\tau_0$, and y — survival probabilities (panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; panel B: (a) – $\mathcal{P}_p(t)$, (b) – $\mathcal{P}_0(t/\gamma)$, (c) – $\mathcal{P}_0(t)$).

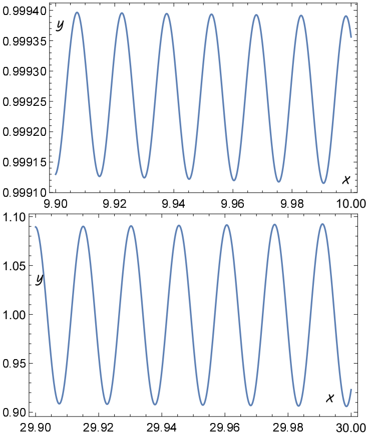


Rysunek: (9) Decay curves obtained for $\omega(\mu)$ given by Eq. (43). Axes: $x = t/\tau_0$, and y — survival probabilities (panel A: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; panel B: $\mathcal{P}_p(t)$; panel C: $\mathcal{P}_0(t/\gamma)$).

Similarly to the case of quantum unstable systems in rest one can calculate the ratio $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$ in the case of moving particles. Results of numerical calculations of this ratio are presented in Figures (10) and (11), and calculations were performed for $\omega(\mu) = \omega_{BW}(\mu)$ and for $\mu_0 = 0$, $m_0/\Gamma_0 = 1000$, $cp/\Gamma_0 \equiv p/\Gamma_0 = 1000$ and $\gamma = \sqrt{2}$.



Rysunek: (10) Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — Ratio of probabilities — Solid line: $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$; Dashed line $\mathcal{P}_0(t/\gamma)/\mathcal{P}_c(t/\gamma)$.



Rysunek: (11) Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — Ratio of probabilities: $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$.

To conclude this discussion, note that within the theory considered the case of $\vec{v} = const$ leads to the wrong result. Simply, if $\vec{v} = const$ then $\gamma = const$ and $\mu\,\gamma_\mu$ in (40) is replaced by $\mu\gamma$ and thus (40) takes the following form in such a case,

$$H|\mu; p\rangle = \mu\gamma|\mu; p\rangle, \tag{44}$$

which leads to the following expression for the amplitude $a_p(t)$:

$$a_p(t) \;=\; \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\mu\gamma\, t}\, d\mu \equiv a_0(\gamma t). \tag{45}$$

This gives the result

$$\mathcal{P}_p(t) = |a_p(t)|^2 \equiv |a_0(\gamma t)|^2 = \mathcal{P}_0(\gamma t),$$

which was never met in experiments.

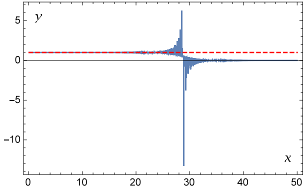
Results published e.g. in [13, 39]

- K. Urbanowski, Decay law of relativistic particles: Quantum theory meets special relativity, *Physics Letters B* **737**, 346, (2014).
- K. Urbanowski, Non–classical behavior of moving relativistic unstable particles, *Acta Physica Polonica B*, **48**, 1411 (2017).

and in other papers were presented in this part of the talk.

- Oscillatory modulated decay laws and the instantaneous energy

From the theoretical and numerical analysis presented earlier in this talk we know that at times $0 < t < \infty$ the oscillatory modulation of the survival probability and instantaneous energy (mass) takes place. The amplitude of these oscillations of the instantaneous energy (mass) is extremely large at transition time region between exponential–like and non–exponential form of the survival amplitude (see also e.g. Fig. (6)):



Rysunek: (12) The instantaneous energy $E_{\phi}(t)$ as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \kappa(t)$, where $\kappa(t)$ is defined by (39); $x = t/\tau_{\phi}$: Time is measured in lifetimes. The horizontal dashed line represents the value of $E_{\phi}(t) = E_0$ $(\mu_{\phi}(t) = m_0)$. $s_R = \frac{E_0 - E_{min}}{\Gamma_0} \equiv \frac{m_0 - \mu_0}{\Gamma_0} = 100$.

In order to observe these effects one needs a very large number of unstable particles. So, it seems that there is a chance to observe this effect or its implications using astrophysical sources of unstable particles emitting huge numbers of them with relativistic or ultra–relativistic velocities in the relation to an external observer. Many of these particles move freely in space with ultrahigh energies. Now the energy conservation together with the fluctuations of the instantaneous energy force these particles to move with the velocity varying in time. As a result such unstable particles, which are charged or have non–zero magnetic moment, have to emit electromagnetic radiation including X– and γ–rays. Details can be found, e.g. here [40]:

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Possible emission of cosmic X- and γ-rays by unstable particles at late times

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- Cosmological implications

Krauss and Dent analyzing a false vacuum decay [38] pointed out that in eternal inflation, many false vacuum regions can survive up to the times much later than times when the exponential decay law approximately holds. Within this scenario $|\phi\rangle = |0\rangle^F$, and $|0\rangle^T$ - are a false and a true vacuum states respectively and $E_0 \equiv E_{qft} = \; ^F\langle 0|H|0\rangle^F$ is the energy of a state corresponding to the false vacuum measured at the initial instant of the decay process and $E_{min} \equiv E_0^T$ is the energy of true vacuum (i.e. the true ground state of the system). Now one can use the property that the energy of the system in the false vacuum state has the following form at asymptotically late times

$$E^F(t) \simeq E_0^T + d_2 \frac{\hbar^2}{t^2} + d_4 \frac{\hbar^4}{t^4} + \cdots \neq E_0^F. \tag{46}$$

Next, if to identify $\rho_{\text{de}}(t_0)$ with the energy E_0 of the unstable system divided by the volume V_0 (where V_0 is the volume of the system at $t = t_0$): $\rho_{\text{de}}(t_0) \equiv \rho_{\text{de}}^{\text{qft}} \stackrel{\text{def}}{=} \rho_{\text{de}}^0 = \frac{E_0}{V_0}$ and $\rho_{\text{bare}} = \frac{E_{\text{min}}}{V_0}$, (where $\rho_{\text{de}}^{\text{qft}}$ is the vacuum energy density calculated using quantum field theory methods), then there is at asymptotically late, post–exponential times,

$$\rho_{\text{de}}(t) = \rho_0^{\text{F}}(t) \simeq \rho_{\text{bare}} + \frac{f_2}{t^2} + \frac{f_4}{t^4} + \cdots, \quad (t \to \infty), \tag{47}$$

where $f_{2k} = f_{2k}^*$. The analogous relation takes place for $\Lambda(t) = \frac{8\pi G}{c^2}\,\rho(t)$, (or $\Lambda(t) = 8\pi G\,\rho(t)$ in $\hbar = c = 1$ units),

$$\Lambda(t) \simeq \Lambda_{\text{bare}} + \frac{\alpha_2}{t^2} + \frac{\alpha_4}{t^4} + \cdots, \quad (t \to \infty). \tag{48}$$

From the above formulae it follows that Λ_{bare} is the limiting value of $\Lambda(t)$ reached when $t \to \infty$.
Note that the form of $\kappa(t)$ (see (39) does not change when one passes from energies $E(t)$, E_0, E_{min} to the above defined energy density ρ and Λ,

$$\kappa(t) = \frac{E(t) - E_{\text{min}}}{E_0 - E_{\text{min}}} \equiv \frac{\frac{E(t)}{V_0} - \frac{E_{\text{min}}}{V_0}}{\frac{E_0}{V_0} - \frac{E_{\text{min}}}{V_0}} = \frac{\rho_{\text{de}}(t) - \rho_{\text{bare}}}{\rho_{\text{de}}^0 - \rho_{\text{bare}}} = \frac{\Lambda(t) - \Lambda_{\text{bare}}}{\Lambda_0 - \Lambda_{\text{bare}}}. \tag{49}$$

The conception presented above was studied and developed, e.g. in [41, 42, 43, 44, 45, 46]:

- K. Urbanowski, Comment on "Late time behavior of false vacuum decay: Possible implications for cosmology and metastable inflating states", *Phys. Rev. Lett.* **107**, 209001, (2011).
- K. Urbanowski, Properties of the false vacuum as the quantum unstable state, *Theor. Math. Phys.* **190**, 458, (2017).
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and in other papers.

- The mass of the system in the unstable state $|\phi\rangle$ is not defined: It can not take the exact value. Unstable system can be characterized by the mass distribution $\omega(\mu)$, the average mass $< m >= \int_{\mu_0}^{\infty} \mu\, \omega(\mu) d\mu$ and by instantaneous mass (energy) $\mu_\phi(t)$ but not by the mass.
- There is no any time interval in which the survival probability (decay) law could be a decreasing function of time of the purely exponential form: Even in the case of the Breit–Wigner model in so–called "exponential regime" the decay curves are oscillatory modulated with smaller or large amplitude of oscillations depending on the parameters of the model.
- At any time interval the instantaneous mass $\mu_\phi(t)$ and instantaneous decay rate $\gamma_\phi(t)$ can not be constant in time.

- In the case of moving relativistic quantum unstable system moving with constant momentum $\vec{p}$, when unstable systems are modeled by the Brei–Wigner mass distribution $\omega(\mu)$, only at times of the order of lifetime τ_0 it can be $\mathcal{P}_p(t) \simeq \mathcal{P}_0(t/\gamma)$ to a better or worse approximation. At times longer than a few lifetimes the decay process of moving particles observed by an observer in his rest system is much slower that it follows from the classical physics relation $\mathcal{P}_p(t) \stackrel{?}{=} \exp\left[-\frac{t}{\gamma}\,\Gamma_0\right]$:

$$\mathcal{P}_p(t) > \mathcal{P}_0(t/\gamma), \quad \text{for} \quad t \gg \tau_0.$$

- In the case of moving relativistic quantum unstable system moving with constant momentum $\vec{p}$ decay curves are also oscillatory modulated but the amplitude of these oscillations is higher than in the case of unstable systems in rest.
- There is a need to test the decay law of moving relativistic unstable system for times much longer than the lifetime

- The experimental verification of deviations from the exponential decay law is still of interest [47, 48]:
  - Alec Cao, Cora J. Fujiwara, Roshan Sajjad, Ethan Q. Simmons, Eva Lindroth and David Weld, Probing Nonexponential Decay in Floquet–Bloch Bands, *Zeitschrift für Naturforschung A*.**75**, 443 —- 448, (2020).
  - Gustav Andersson, Baladitya Suri, Lingzhen Guo, Thomas Aref and Per Delsing, Non-exponential decay of a giant artificial atom, *Nature Physics*,**5**, 1123 — 1127, (2019).
- GSI anomaly: Recent experiments and analysis of experimental data does not confirm a oscillatory modulation of decay curve [49]:
  - F.C.Ozturk. *et al*, New test of modulated electron capture decay of hydrogen–like ^{142}Pm ions: Precision measurement of purely exponential decay, *Physics Letters B* **797**, 134800, 2019.

Research in progress:

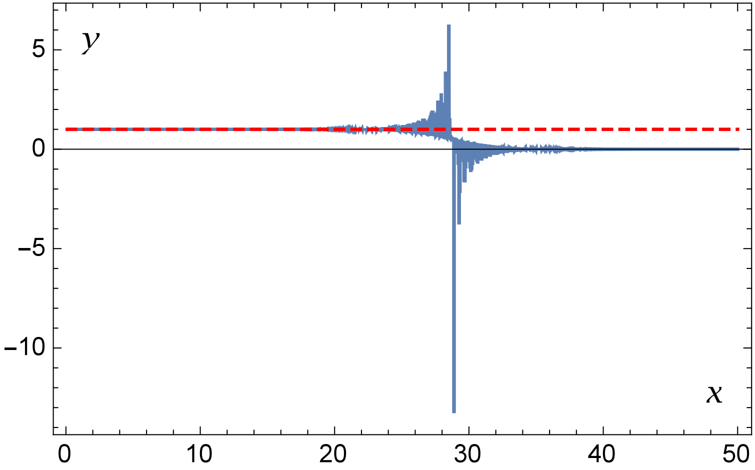
The reasonable approximation is to assume that Λ_{bare} equals to the present measured value of the cosmological constant. On the other hand there is $\Lambda_0 \simeq 10^{120}\Lambda_{\text{bare}}$, where $\Lambda_0 = \Lambda_{\text{qft}} = \frac{8\pi G}{c^2}\,\rho^0_{\text{de}}$ is the result obtained by means methods of the quantum field theory. The great problem is to explain this difference. It seems that the following observation can help to solve this problem: There is

$$\Lambda(t) = \Lambda_{bare} + (\Lambda_0 - \Lambda_{bare})\,\kappa(t), \tag{50}$$

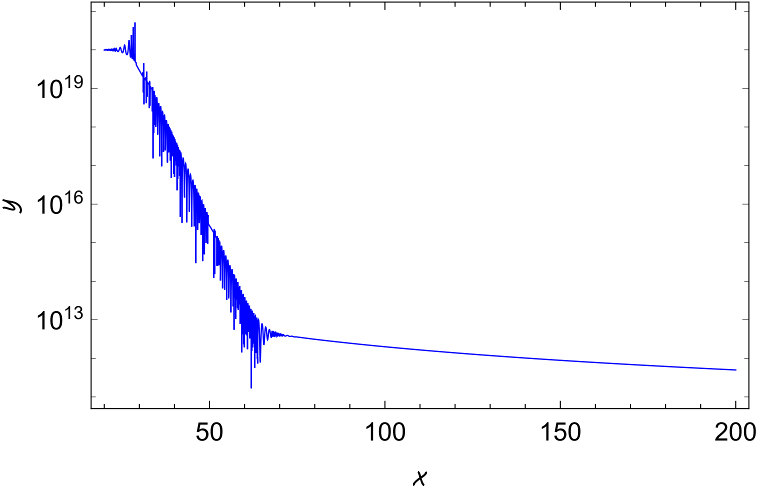
or

$$\frac{\Lambda(t)}{\Lambda_{bare}} = \left(\frac{\Lambda_0}{\Lambda_{bare}} - 1\right)\,\kappa(t) \equiv \left(\frac{E_0}{E_{min}} - 1\right)\,\kappa(t), \tag{51}$$

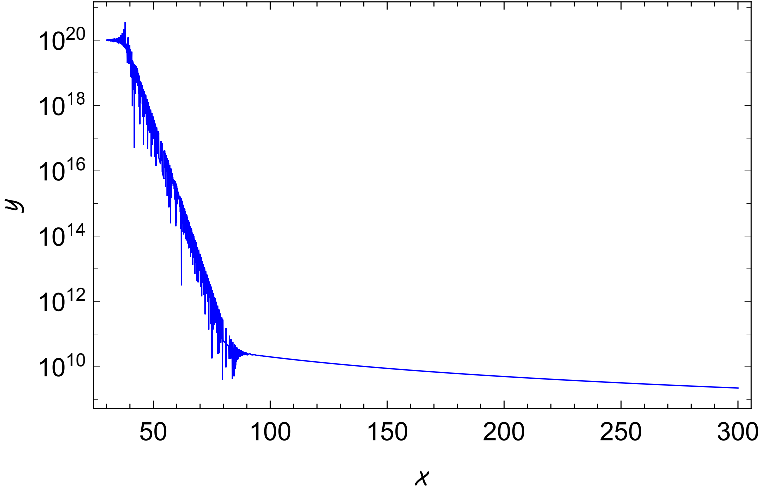
(see formulae (39), (49) for $\kappa(t)$). Using (51) and assuming values of the ratios $\frac{E_0}{E_{min}}$, $s_R = \frac{E_0 - E_{min}}{\Gamma_0}$ one can find suitable parameters of the density $\omega(E)$ and then to find a class of Hamiltonians H being able to reproduce desirable effect of the reduction Λ_0 to Λ_{bare}. Partial results are presented below in Figs (14), (15).



Rysunek: (13) The instantaneous mass $\mu_\phi(t)$ as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \kappa(t)$, where $\kappa(t)$ is defined by (39); $x = t/\tau_\phi$: Time is measured in lifetimes. The horizontal dashed line represents the value of $\mu_\phi(t) = m_0$. $s_R = \frac{m_0 - \mu_0}{\Gamma_0} = 100$.



Rysunek: (14) The instantaneous energy E(t) (mass $\mu_\phi(t)$) as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \left|\frac{E(t)}{E_{min}}\right| \equiv \left|\frac{\Lambda(t)}{\Lambda_{bare}}\right|$, $x = t/\tau_\phi$: Time is measured in lifetimes. The case $\frac{E_0}{E_{min}} \equiv \frac{m_0}{\mu_0} = \frac{\Lambda_{qft}}{\Lambda_{bare}} = 10^{20}$, $s_R = \frac{m_0 - \mu_0}{\Gamma_0} = 100$.



Rysunek: (15) The instantaneous energy E(t) (mass $\mu_\phi(t)$) as a function of time obtained for $\omega_{BW}(\mu)$. Axes: $y = \left|\frac{E(t)}{E_{min}}\right| \equiv \left|\frac{\Lambda(t)}{\Lambda_{bare}}\right|$, $x = t/\tau_\phi$: Time is measured in lifetimes. The case $\frac{E_0}{E_{min}} \equiv \frac{m_0}{\mu_0} = \frac{\Lambda_{qft}}{\Lambda_{bare}} = 10^{20}$, $s_R = \frac{m_0 - \mu_0}{\Gamma_0} = 1000$.

Thank you for your attention

Appendix

Let $\Lambda_{p,\mu}$ be the Lorentz transformation from the reference frame $\mathcal{O}_0$, where the momentum of the unstable particle considered is zero, $\vec{p} = 0$, into the frame $\mathcal{O}_p$ where the momentum of this particle is $\vec{p} \equiv (p, 0, 0) \neq 0$ and $p \geq 0$, or, equivalently, where its velocity equals $\vec{v} = \vec{v}_{p,\mu} \equiv \frac{\vec{p}}{\mu\,\gamma_\mu}$, (where μ is the rest mass and $\gamma_\mu \equiv \sqrt{p^2 + (\mu)^2}/\mu$).

In this case the corresponding 4–vectors are:

$$\wp = (E/c, 0, 0, 0) \equiv (\mu, 0, 0, 0) \in \mathcal{O}_0$$

within the considered system of units, and

$$\wp' = (E'/c, p, 0, 0) \equiv (E', p, 0, 0) = \Lambda_{p,\mu}\, \wp \in \mathcal{O}_p.$$

There is

$$\wp' \cdot \wp' \equiv (\Lambda_{p,\mu}\, \wp) \cdot (\Lambda_{p,\mu}\, \wp) = \wp \cdot \wp$$

in Minkowski space, which is an effect of the Lorentz invariance. (Here the dot "$\cdot$" denotes the scalar product in Minkowski space). Hence, in our case:

because

$$\wp \cdot \wp \equiv \mu^2$$

and thus

$$(E')^2 \equiv (E'(\mu, p))^2 = p^2 + \mu^2.$$

Another way to find $E'(\mu, p)$ is to use the unitary representation, $U(\Lambda_{p,\mu})$, of the transformation $\Lambda_{p,\mu}$, which acts in the Hilbert space $\mathcal{H}$ of states $|\phi\rangle \equiv |\phi; 0\rangle, |\phi_p\rangle \in \mathcal{H}$.

One can show that the vector $U(\Lambda_{p,\mu})|\mu;0\rangle$ is the common eigenvector for operators H and **P**, that is that there is

$$|\mu;p\rangle \equiv U(\Lambda_{p,\mu})|\mu;0\rangle$$

(see, eg. [16]). Indeed, taking into account that operators H and **P** form a 4–vector P_ν,

$$P_\nu = (P_0, \mathbf{P}) \equiv (P_0, P_1, 0, 0), \;\; \text{and} \;\; P_0 \equiv H,$$

we have

$$U^{-1}(\Lambda_{p,\mu}) P_\nu U(\Lambda_{p,\mu}) = \Lambda_{p,\mu;\,\nu\lambda}\, P_\lambda,$$

where $\nu, \lambda = 0, 1, 2, 3$ (see, e.g., [16], Chap. 4). From this general transformation rule it follows that

$$\begin{aligned} U^{-1}(\Lambda_{p,\mu}) P_0 U(\Lambda_{p,\mu}) &= \gamma_\mu\,(P_0 + v_\mu\, P_1) \\ &\equiv \gamma_\mu (H + v_\mu\, P_1), \end{aligned} \tag{52}$$

and

$$\begin{aligned} U^{-1}(\Lambda_{p,\mu}) P_1 U(\Lambda_{p,\mu}) &= \gamma_\mu\,(v_\mu P_0 + P_1) \\ &\equiv \gamma_\mu (v_\mu H + P_1), \end{aligned} \tag{53}$$

Based on the relation (52), one can show that that vectors $U(\Lambda_{p,\mu})|\mu;0\rangle$ are eigenvectors for the Hamiltonian H. There is

$$\begin{aligned} H\,U(\Lambda_{p,\mu})|\mu;0\rangle &= U(\Lambda_{p,\mu})\,U^{-1}(\Lambda_{p,\mu})\,H\,U(\Lambda_{p,\mu})|\mu;0\rangle \\ &= \gamma_\mu\,U(\Lambda_{p,\mu})\,(H\,+\,v_\mu\,P_1)\,|\mu;0\rangle. \end{aligned} \tag{54}$$

The Lorentz factor γ_μ corresponds to the rest mass μ being the eigenvalue for the vector $|\mu;0\rangle$. There are $\gamma_\mu \neq \gamma_{\mu'}$ and $v_\mu \neq v_{\mu'}$ for $\mu \neq \mu'$. From (5), (8) it follows that $P_1\,|\mu;0\rangle = 0$ for $p = 0$, which means that using (9) the relation (54) can be rewritten as follows

$$H\,U(\Lambda_{p,\mu})|\mu;0\rangle = \mu\gamma_\mu\,U(\Lambda_{p,\mu})|\mu;0\rangle. \tag{55}$$

Taking into account the form of the γ_μ forced by the condition $p = const$ one concludes that in fact the eigenvalue found, $\mu\gamma_\mu$, equals

$$\mu\gamma_\mu \equiv \sqrt{p^2 + \mu^2}.$$

This is exactly the same result as that at the conclusion following from the Lorentz invariance mentioned earlier:

$$E'(\mu,p) = \sqrt{p^2 + \mu^2},$$

which shows that the above considerations are self–consistent.

Similarly one can show that vectors $U(\Lambda_{p,\mu})|\mu;0\rangle$ are the eigenvectors of the momentum operator **P** for the eigenvalue $\mu\gamma_\mu\, v_\mu \equiv p$, that is that

$$U(\Lambda_{p,\mu})|\mu;0\rangle \equiv |\mu;p\rangle.$$

Using (53) one finds

$$\begin{aligned} P_1\, U(\Lambda_{p,\mu})|\mu;0\rangle &= U(\Lambda_{p,\mu})\, U^{-1}(\Lambda_{p,\mu})\, P_1\, U(\Lambda_{p,\mu})|\mu;0\rangle \\ &= \gamma_\mu\, U(\Lambda_{p,\mu})\,(v_\mu\, H \,+\, P_1)\,|\mu;0\rangle. \end{aligned} \tag{56}$$

Again taking into account properties

$$P_1\,|\mu;0\rangle = 0 \;\text{ and }\; H|\mu;0\rangle = \mu|\mu;0\rangle$$

we conclude that

$$\begin{aligned} P_1\, U(\Lambda_{p,\mu})|\mu;0\rangle &= \mu\gamma_\mu v_\mu\, U(\Lambda_{p,\mu})|\mu;0\rangle \\ &\equiv p\, U(\Lambda_{p,\mu})|\mu;0\rangle, \end{aligned} \tag{57}$$

that is that $U(\Lambda_{p,\mu})|\mu;0\rangle \equiv |\mu;p\rangle$ which was to show.

Thus finally we come to desired results:

$$H|\mu; p\rangle = \sqrt{p^2 + \mu^2}\,|\mu; p\rangle \tag{58}$$

which replaces Eq. (6).

The moving quantum unstable particle ϕ with constant momentum, $\vec{p}$, can be modeled analogously as the quantum unstable system in the rest frame (when $\vec{p} = 0$) as the following wave–packet $|\phi_p\rangle$,

$$|\phi_p\rangle = \int_{\mu_0}^{\infty} c(\mu)\,|\mu; p\rangle\, dm, \tag{59}$$

where expansion coefficients $c(\mu)$ are functions of the mass parameter μ, that is of the rest mass μ, which is Lorentz invariant and therefore the scalar functions $c(\mu)$ of μ are also Lorentz invariant and are the same as in the rest reference frame $\mathcal{O}_0$.

Now using (40) and the equation (41) we obtain the final, required relation for the amplitude $a_p(t)$ (see [10, 11, 17]),

$$a_p(t) = \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\mu\gamma_\mu t}\, d\mu \qquad (60)$$

$$\equiv \int_{\mu_0}^{\infty} \omega(\mu)\, e^{-\,i\sqrt{p^2+\mu^2}\;t}\, d\mu. \qquad (61)$$

This is the place when it should be explained why the Lorentz factor γ_μ is used in (60) (and earlier in relations (52) — (57) instead of $\gamma_\mu = \gamma$. In the rest reference frame the unstable quantum system is modeled as the wave packet given by the relation (11), that is as the following wave–packet $|\phi_0\rangle \equiv |\phi_{\vec{p}=0}\rangle \stackrel{\text{def}}{=} |\phi\rangle$,

$$|\phi_0\rangle \equiv |\phi\rangle = \int_{\mu_0}^{\infty} c(\mu)\,|\mu;0\rangle\, d\mu, \qquad (62)$$

Let us choose some eigenvalues of the spectrum $\sigma_c(H)$ of H:

$$\mu_1 < \mu_2 < \ldots < \mu_k < \ldots < \mu_n \in \sigma_c(H) = [\mu_0, \infty),$$

These eigenvalues are connected with corresponding eigenvectors $|\mu_k; 0\rangle$ of H as follows:

$$\begin{array}{ccccccc} & & \mathcal{O}_0 & & \mathcal{O}_p & & \\ \mu_1 & \leftrightarrow & |\mu_1; 0\rangle & \overset{U(\Lambda_{p;\mu_1})}{\longmapsto} & |\mu_1; p\rangle & \leftrightarrow & p = \mu_1 \gamma_{\mu_1} v_{\mu_1} \\ \mu_2 & \leftrightarrow & |\mu_2; 0\rangle & \overset{U(\Lambda_{p;\mu_2})}{\longmapsto} & |\mu_2; p\rangle & \leftrightarrow & p = \mu_2 \gamma_{\mu_2} v_{\mu_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_k & \leftrightarrow & |\mu_k; 0\rangle & \overset{U(\Lambda_{p;\mu_k})}{\longmapsto} & |\mu_k; p\rangle & \leftrightarrow & p = \mu_k \gamma_{\mu_k} v_{\mu_k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mu_n & \leftrightarrow & |\mu_n; 0\rangle & \overset{U(\Lambda_{p;\mu_n})}{\longmapsto} & |\mu_n; p\rangle & \leftrightarrow & p = \mu_n \gamma_{\mu_n} v_{\mu_n}, \end{array} \tag{63}$$

(Let us recall that $p = const.$). As it is seen from the above analysis each vector $|\mu; 0\rangle$ numbered by $\mu \in \sigma_c(H)$ can be transformed correctly in the vector $|\mu; p\rangle$ connected with the reference frame $\mathcal{O}_p$ only if one takes into account that every point μ from the spectrum of H, considered as the rest mass, has the "own" Lorentz factor

$$\gamma = \frac{\sqrt{\mu^2 + p^2}}{\mu} \equiv \gamma_\mu.$$

(The reference frame $\mathcal{O}_p$ was defined by condition $\vec{p} = const \neq 0$, and $\mathcal{O}_0 = \mathcal{O}_{p=0}$). In other words it is impossible to realize the above transformations of vectors $|\mu; 0\rangle$ assigned to a reference frame $\mathcal{O}_0$ to the reference frame $\mathcal{O}_p$ if $\vec{v} = const$.

The above derivation of the expression for $a_p(t)$ is similar to that of [13]. It is based on [16] and it is reproduced here for the convenience of listeners. This is a shortened and slightly changed, simplified version of the considerations presented in [10] and mainly in [11].

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